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THEORY OF FINITE SYSTEMS OF PARTICLES

II. SCATTERING THEORY

BY

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Synopsis

A scattering theory is presented for a non-relativistic system consisting of a finite number of particles with local two-body interactions. The behaviour of the system is studied from the point of view of the theory of Hilbert space. The formalism aims at expressing the scattering of wave-packets in terms of the resolvent of the Hamiltonian.

The starting-point is Jauch's theory of wave-operators, which is summarized. It is explained that this theory is particularly well suited for describing multi-channel processes. It permits an unambiguous definition of reaction channels and is sufficiently general to discuss the scattering, both elastic and inelastic, of any finite number of particles or bound fragments.

In Jauch's theory, there is a condition on the time development of the system. In order that there exist wave-operators, the system must split into mutually independent fragments as the time tends to ∞ or $-\infty$. In the present paper, this condition is translated into a condition on the two-body interactions. It is shown that for the existence of wave-operators it is sufficient, roughly speaking, if the two-body interactions are locally square-integrable and at large distances tend to 0 faster than the Coulomb interaction. This result applies to general multi-channel processes.

If the interaction satisfies sufficient conditions, the wave-operators can easily be related to the resolvent. This is done with the spectral theory of self-adjoint operators. It is found, however, that the theory of the scattering operators still meets with practical difficulties. These have to do with repeated limits that cannot be interchanged. To obtain workable expressions for the scattering operators, some further conditions are imposed upon the interaction. Also, the discussion is restricted to wave-packets that satisfy certain smoothness criteria. The scattering of smooth wave-packets is described by the limit of a sequence of linear functionals. In this sequence, the wave-function plays the part of a test-function. Each functional contains the resolvent for complex energies in the neighbourhood of the continuous spectrum. The limit refers to the energy tending to real values. The limiting behaviour is discussed in detail.

For systems with spherically symmetric two-body interactions, particular attention is devoted to scattering events in which both in the distant past and in the remote future there are only two fragments. For such events, it is shown how from the general expression for the scattering of a wave-packet, one can extract a scattering matrix the elements of which are functions of a real energy-parameter. Also, a study is made of the scattering of a collimated beam. This is not described by a plane wave, but by a statistical mixture of wave-packets. For the total scattering intensity to be finite, it is sufficient if the interaction between scattered fragments is locally square-integrable and at large distances tends to 0 faster than the inverse distance squared. Under this condition it is possible to define the amplitudes for scattering through a fixed angle. For real energies, these quantities are discussed in detail. In particular, it is shown how they are related to the resolvent and to sums of scattering-matrix elements. In a forthcoming paper the present results will be used to continue the scattering amplitudes into the complex plane, and to investigate their analytic properties.

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2.1. Introduction

2.1.1. General outline

In a previous paper with the subtitle "The Green Function" (1), a study was made of the resolvent operator for a system consisting of any finite number of particles. This operator was considered for complex energies not in the continuous spectrum of the Hamiltonian. Under the assumption that in the system there are only square-integrable local two-body interactions, it was shown that the resolvent is an integral operator the kernel of which can be evaluated explicitly. The kernel in question was called the Green function. In the energy plane cut along the real axis from some point M to ∞, the resolvent is analytic, regular except for possible poles on the real axis. If there are poles, these are located at the energies of bound states, the corresponding eigenfunctions following from the residues of the resolvent. Since in the cut energy plane the resolvent can be evaluated explicitly, the bound states of the system considered have thus been made accessible to further investigation. By contrast, the previous paper does not give any information on the structure of the continuous spectrum, nor, in fact, does it determine its location. Since obviously there is an intimate relationship between the existence of a continuous spectrum and the occurrence of scattering phenomena, we therefore continue our investigation of finite systems of particles with a paper on the theory of scattering.

In a scattering process there is a number of fragments which in the distant past were very far apart and consequently behaved as if they were free. In the course of time, the fragments approach each other, and a collision takes place. This may cause the fragments to change their velocities. It may also give rise to reactions. After the collision there will in any case emerge a number of fragments. In the remote future these will be free again. It is the object of the scattering theory, firstly, to say what states can occur as initial and final states in a scattering event and, secondly, to evaluate the probability for transitions between these states. In the course of years numerous papers have been devoted to this subject. However, in none of these did we find a rigorous treatment that takes us from first principles to explicit expressions for observable quantities related to the continuous spectrum.

The formalism of the present paper has as its starting-point a theory due mainly to Jauch (2, 3), which expresses in a precise mathematical form the essential features of a scattering process. Jauch's theory is concerned with the time development of suitably chosen wave-packets, which are required to tend to free packets as the time tends to ∞ or $-\infty$. The requirement that there should be limits makes it possible to define the wave-operators and, subsequently, the scattering operators, which determine the transition probabilities. For the simple case of one-channel scattering, this is summarized in section 2.2.

It is one of the beautiful features of Jauch's theory that it makes possible an unambiguous description of multi-channel processes. This is explained in section 2.3. From this section it will become clear that the theory is a considerable improvement on the usual heuristic scattering formalism, in which there are always difficulties associated with the possibility for reactions to take place. In particular, it is no longer necessary to restrict the discussion to processes in which the total scattering system is not separated into more than two fragments. Neither need there be an exterior region in configuration space in which there is no interaction between the fragments. The most important improvement, however, is concerned with the channel concept. In the usual formalism, reaction channels are defined via a discussion of the wave-function in the external region. In this region an expansion is made in terms of channel wave-functions which are, however, not strictly orthogonal. As a result the decomposition into channels is not really unique. If the system is in channel a, this does not in general exclude its being in channel b. This ambiguity is completely avoided in the formalism to be summarized in section 2.3, the point being that in Jauch's theory one concentrates on the ergodic properties of the system, rather than on its asymptotic behaviour in configuration space.

Sections 2.2 and 2.3 are completely formal in the sense that it is simply assumed that the properties of the system are such that, as the time tends to ∞ or $-\infty$, the wave-function tends to well-behaved limits. This is equivalent to the assumption that there exist wave-operators and scattering operators. As it stands this requirement is fairly abstract. However, for one-channel problems it was shown by Cook(4), Jauch and Zinnes(5), and Kuroda(6) that this point can be traced back to the interaction in the system. This is discussed in section 2.4, in which we also treat the multi-channel case. It is found that for the existence of the scattering operators it is sufficient, roughly speaking, that the interaction between scattered fragments is locally square-integrable and at large distances falls off more rapidly than the Coulomb interaction. For a large class of interactions it is shown that the set of states that can occur as initial or final states in a scattering event is uniquely determined. This means that within the framework of the present formalism there is one and only one way in which channels can be defined. The channels are mutually orthogonal, according to section 2.3.

In section 2.5 the wave-operators are expressed in terms of the resolvent. This is done with the spectral theory developed in section 1.4 of our previous paper(1). Since by the previous paper the resolvent can be evaluated explicitly, section 2.5 makes it possible, in principle, to compute the wave-operators.

When in section 2.6 we try to extend the methods of section 2.5 to the scattering operators, an unexpected difficulty presents itself. There appears a repeated limit which in general is completely unmanageable. It is found, however, that, if the relative motion of the scattered fragments and their mutual interaction are what we call admissible, one limit can be performed explicitly. There then remains a limit which involves the resolvent and essentially refers to the energy of the system approaching the continuous spectrum through non-real values. This limit exists owing to the fact that, as the time tends to ∞ or $-\infty$, the system splits into independent fragments sufficiently rapidly. This splitting, in turn, is due to the interaction between the fragments decreasing sufficiently rapidly as their distance increases. We thus see that there is an intimate relationship between the behaviour of the resolvent in the neighbourhood of the continuous spectrum, the time development of scattered wave-packets, and the properties of the interaction. In some of our formulas there is an analogy with the work of LIPPMANN and SCHWINGER(7) and GELL-MANN and GOLDBERGER(8), but the general point of view is entirely different.

The requirement that the interaction and the relative motion of the scattered fragments be admissible is discussed in sections 2.6.3 to 2.6.6. For the interaction, sufficient conditions are found which are only slightly more restrictive than the mere existence of the scattering operators. For the relative motion, wave-functions are chosen which in momentum space are smooth and vanish outside bounded regions. The more detailed results of the present paper all refer to such wave-functions. The scattering of smooth wave-packets is described by the limit of a sequence of linear functionals. In the terminology of the theory of distributions, the wave-function plays the role of the test-function. Each member of the sequence contains the resolvent for complex energies in the neighbourhood of the continuous spectrum. In the limit the energy tends to the real axis, as mentioned above. It is obvious that the formalism must yield the conservation of energy during the scattering process. This comes out in a natural way, without the intermediary of δ-functions. Also, there are no normalization difficulties.

The formulas of section 2.6 apply to the scattering of any number of fragments. It is discussed in section 2.7 that considerable simplifications become possible if we restrict ourselves to scattering events in which both in the distant past and in the remote future there are only two fragments. For systems with spherically symmetric two-body interactions, particular attention is devoted to the scattering of partial waves. It is shown how this can be described with the help of a matrix, the elements of which are functions of a real energy-parameter. Under suitable conditions this $\mathcal{S}$-matrix is unitary and symmetric.

Whereas the $\mathcal{S}$-matrix refers to one single wave-packet, section 2.8 discusses the scattering of what we call a beam. By this we mean a certain statistical mixture of wave-packets. By analogy with a plane wave, a beam can be decomposed into a sum of partial waves. The requirement that this sum be convergent imposes the restriction that, as the distance between scattered fragments increases, their interaction decreases faster than the inverse distance squared. If this condition is fulfilled, it is possible to define the amplitudes for scattering through a fixed angle. These determine the cross section. A beam as defined here provides a good description of a collimated stream of projectiles. It is, in fact, much more appropriate than a plane wave. Mathematically a beam can be handled more easily. Since it consists of a mixture of wave-packets, it is also more acceptable physically than a plane wave.

Qualitatively, the result that there is an $\mathcal{S}$-matrix and scattering amplitudes is what one expects from more heuristic theories. It must be remarked, however, that a correct definition of these quantities requires a careful handling of limits and integrals that cannot be interchanged. In the present treatment an element of the $\mathcal{S}$-matrix comes out as the derivative of the limit of a sequence of integrals. A scattering amplitude is defined as the sum of a series which converges in mean square. With the help of section 2.6 the $\mathcal{S}$-matrix and the scattering amplitudes are related to the resolvent. Mathematically this is also a subtle affair.

The insight we have gained into the limiting properties of the resolvent in the neighbourhood of the continuous spectrum is very useful for further research. In a forthcoming paper it will make it possible to consider the $\mathcal{S}$-matrix elements and the scattering amplitudes as the boundary values of analytic functions that depend on a complex energy. It will be shown that the boundary behaviour is sufficiently smooth for these analytic functions to satisfy dispersion relations. Because of the dispersion relations, there is also a parameter expansion for the $\mathcal{S}$-matrix. This nicely describes the qualitative features of the scattering. In particular, it exhibits resonances against a background of direct reactions.

2.1.2. Notation and basic assumptions

In the following our previous paper(1) is denoted by I. We use the same notation. Since in I all sections and formulas were given numbers beginning with 1, they can simply be referred to by the original numbers.

We recall from I that the present investigation is concerned with systems consisting of n distinguishable particles, the Hamiltonian being derived from

$$H'(\boldsymbol{X}) = -\sum_{i=1}^{n} \frac{1}{2\,m_i} \Delta(\boldsymbol{X}_i) + \sum_{i<j} V_{ij}(\boldsymbol{X}_i - \boldsymbol{X}_j) \tag{2.1.1}$$

(cf. eq. (1.1.1)). As was the case in I, it is useful to split off the motion of the centre of mass. If this is done through the introduction of new coordinates $\boldsymbol{x}$ according to eq. (1.2.1), the operator for the relative motion takes the form

$$H'(\boldsymbol{x}) = -\sum_{i=1}^{n-1} \Delta(\boldsymbol{x}_i) + \sum_{i<j} V_{ij}\left(\sum_{k=1}^{j-1} c_{ij}^{k} \boldsymbol{x}_k\right), \tag{2.1.2}$$

with certain constants c (cf. eq. (1.2.4)).

It was explained in section 1.2.2 that, under suitable assumptions on the functions V_{ij}, there is a unique way of extending $H'(\boldsymbol{x})$ to a self-adjoint operator in the Hilbert space of square-integrable functions of $\boldsymbol{x}$. This self-adjoint extension is taken as the Hamiltonian. It is denoted by H or $H(\boldsymbol{x})$. Its domain is denoted by $\mathfrak{D}(H)$. If there is no interaction, H reduces to the operator H_0 with domain $\mathfrak{D}(H_0)$.

For the Hilbert space of square-integrable functions of $\boldsymbol{x}$ we use the notation $\mathfrak{L}^2$ or $\mathfrak{L}^2(\boldsymbol{x})$. If f and g are any two functions in $\mathfrak{L}^2$, the norm of f is denoted by $||f||$, the inner product of f and g by (g,f),

$$||f|| = [\int|f(\boldsymbol{x})|^2 d\boldsymbol{x}]^{\frac{1}{2}}, \qquad (g,f) = \int \bar{g}(\boldsymbol{x})f(\boldsymbol{x})d\boldsymbol{x}. \tag{2.1.3}$$

It is assumed throughout the present paper that the operator $H'(\boldsymbol{x})$ is essentially self-adjoint, i. e. that it has one and only one self-adjoint extension. This point was discussed in great detail by KATO(9). It was shown first that $H_0'(\boldsymbol{x})$ is essentially self-adjoint, the domain $\mathfrak{D}(H_0)$ of the self-adjoint extension H_0 consisting of all functions $f(\boldsymbol{x})$ in $\mathfrak{L}^2(\boldsymbol{x})$ whose Fourier transforms $\hat{f}(\boldsymbol{k})$ are such that $|\boldsymbol{k}|^2\hat{f}(\boldsymbol{k})$ belongs to $\mathfrak{L}^2(\boldsymbol{k})$. Next it was shown by KATO that for $H'(\boldsymbol{x})$ to be essentially self-adjoint, it is sufficient if there are constants α and β, with $\alpha < 1$, such that, for every f in $\mathfrak{D}(H_0)$, the quantity $V_{ij}f$ satisfies

$$\sum_{i<j} ||V_{ij}f|| < \alpha||H_0 f|| + \beta||f|| \tag{2.1.4}$$

(cf. eq. (1.7.14)). If this condition is fulfilled, we have

$$\mathfrak{D}(H) = \mathfrak{D}(H_0), \qquad H = H_0 + \sum_{i<j} V_{ij} \tag{2.1.5}$$

(cf. eq. (1.2.9)). In the present paper it is assumed throughout that eqs. (2.1.4) and (2.1.5) are satisfied.

It follows from the work of KATO(9) and STUMMEL(10) that eq. (2.1.4) holds true for a large class of interactions. Obviously it is fulfilled whenever V_{ij} is bounded. It is also satisfied if

$$\int [V_{ij}(\boldsymbol{X})]^2(1 + |\boldsymbol{X} - \boldsymbol{Y}|)^{-1+\zeta} d^3\boldsymbol{X} < \text{const.} \tag{2.1.6}$$

for some ζ with $0 < \zeta < 1$, and every $\boldsymbol{Y}$. If this relation holds true, the constant α in eq. (2.1.4) may be chosen as close to 0 as we like.

Equation (2.1.6) gives a typical condition on V_{ij} that is often imposed from section 2.4 onwards. In later sections some further restrictions are required. These are indicated as the need arises.

2.2. The scattering of two particles

2.2.1. The time development of the system

As a simple example of a system in which scattering can take place, we consider two particles the interaction of which depends only on their mutual distance. For the Hamiltonian of the relative motion we write H, the corresponding Hamiltonian for the system without interaction being denoted by H_0.

If in the Schrödinger representation the wave-function for the relative motion is equal to f_+ at time $t = 0$, it takes the form $\exp(-iHt)f_+$ at time t. Here f_+ must belong to the Hilbert space $\mathfrak{L}^2$. The operator $\exp(-iHt)$ can most easily be defined through the spectral theory given in section 1.4.

We imagine that the time development of the system is such that in the distant past the two particles were very far apart, and that they behaved approximately as free particles, according to a wave-function $\exp(-iH_0t)f$, with some f in $\mathfrak{L}^2$. That is, we assume that there are functions f_+ and f such that

$$\lim_{t\to-\infty}(e^{-iHt}h,\ e^{-iHt}f_+ - e^{-iH_0t}f) = 0 \tag{2.2.1}$$

for every h in $\mathfrak{L}^2$. A second assumption is that in the course of time the norm of the wave-function does not change,

$$||f_+|| = ||f||. \tag{2.2.2}$$

It follows from eqs. (2.2.1) and (2.2.2) that

$$\left.\begin{aligned} &\lim_{t\to-\infty}||f_+ - e^{iHt}e^{-iH_0t}f||^2\\ &= \lim_{t\to-\infty}\left[||f_+||^2 + ||f||^2 - (e^{-iHt}f_+, e^{-iH_0t}f) - (e^{-iH_0t}f, e^{-iHt}f_+)\right]\\ &= ||f||^2 - ||f_+||^2 = 0. \end{aligned}\right\} \tag{2.2.3}$$

In other words, we assume in fact that there is a function f_+ which is the limit in mean of the sequence $\exp(iHt)\exp(-iH_0t)f$,

$$f_+ = \underset{t\to-\infty}{\text{l.i.m.}}\, e^{iHt}e^{-iH_0t}f. \tag{2.2.4}$$

With a view to discussing the behaviour of the system in the remote future, we likewise assume that there is a funtion f_- such that

$$f_- = \underset{t\to\infty}{\text{l.i.m.}}\, e^{iHt}e^{-iH_0t}f. \tag{2.2.5}$$

For future reference it is convenient to define

$$\Omega(t) = e^{iHt}e^{-iH_0t}. \tag{2.2.6}$$

If f is such that $\Omega(t)f$ has a limit as t tends to $-\infty$, it is not obvious that there is also a limit as t tends to ∞. However, in all practical cases in which we are interested in the following, it turns out that if one limit exists, so does the other. For simplicity we therefore assume that this is so from the outset. While there is a feeling that this point is related to invariance under time reversal (JAUCH (2) footnote p. 136), it seems that it is not well understood. However this may be, let us denote the set of functions f for which both limits exist by $\mathfrak{C}$. Then it is not difficult to see that $\mathfrak{C}$ is a closed set. For let us choose a sequence f_N in $\mathfrak{C}$ which tends in mean to some function f in $\mathfrak{L}^2$. Then we want to prove that for f the limits (2.2.4) and (2.2.5) exist, so that it belongs to $\mathfrak{C}$. For this it is sufficient to show that, given a positive δ, there is a number T such that

$$||\Omega(s)f - \Omega(t)f|| < \delta \qquad (s,t < -T;\; s,t > T). \tag{2.2.7}$$

If eq. (2.2.7) is satisfied, it follows from the fact that the space $\mathfrak{L}^2$ is closed, that there must be a function f_+ in $\mathfrak{L}^2$ such that, as t tends to $-\infty$, the sequence $\Omega(t)f$ tends in mean to f_+. And similarly for f_-. Hence f belongs to $\mathfrak{C}$, and $\mathfrak{C}$ is closed.

To check eq. (2.2.7), we write

$$\left.\begin{aligned} ||\Omega(s)f - \Omega(t)f|| &= ||[\Omega(s) - \Omega(t)](f - f_N) + [\Omega(s) - \Omega(t)]f_N|| \\ &\le ||[\Omega(s) - \Omega(t)](f - f_N)|| + ||\Omega(s)f_N - f_{N\pm}|| + ||\Omega(t)f_N - f_{N\pm}|| \\ &< 2||f - f_N|| + ||\Omega(s)f_N - f_{N\pm}|| + ||\Omega(t)f_N - f_{N\pm}||, \end{aligned}\right\} \tag{2.2.8}$$

$f_{N\pm}$ being the limit of $\Omega(t)f_N$ as t tends to $\mp\infty$. By choosing first N sufficiently large, next s and t sufficiently small, c. q. sufficiently large, the right-hand side of eq. (2.2.8) can be made less than δ. Hence eq. (2.2.7) is satisfied, and our assertion is proved.

In the following the set of functions f_+ which are limits in the sense of eq. (2.2.4) is denoted by $\mathfrak{R}_+$, the corresponding set of functions f_- is denoted by $\mathfrak{R}_-$. Taking into account the fact that $\mathfrak{C}$ is closed, it is easily shown that the sets $\mathfrak{R}_\pm$ are also closed.

2.2.2. The wave-operators

Since the set $\mathfrak{C}$ is closed, every function h in $\mathfrak{L}^2$ can uniquely be decomposed according to

$$h = f + g, \quad f \in \mathfrak{C}, \quad g \perp \mathfrak{C}, \tag{2.2.9}$$

where $g \perp \mathfrak{C}$ means that g belongs to the orthogonal complement of $\mathfrak{C}$. With the help of this decomposition, we now introduce operators $\Omega_\pm$ defined by

$$\Omega_\pm h = \underset{t \to \mp\infty}{\text{l.i.m.}}\, \Omega(t)f. \tag{2.2.10}$$

These operators are called wave-operators. They are bounded operators with domains $\mathfrak{L}^2$ and ranges $\mathfrak{R}_\pm$. The wave-operators have uniquely determined adjoints $\Omega_\pm^*$, which satisfy

$$(k,\Omega_\pm^* h) = (\Omega_\pm k, h) \tag{2.2.11}$$

for every h and k in $\mathfrak{L}^2$. It follows from eq. (2.2.11) that $\Omega_\pm^* h = 0$ whenever $h \perp \mathfrak{R}_\pm$. Also, $(k,\Omega_\pm^* h)$ vanishes if $k \perp \mathfrak{C}$, by eqs. (2.2.10) and (2.2.11). Hence, if $\Omega_\pm^* h$ does not vanish, it belongs to $\mathfrak{C}$.

We now want to show that the operator $\Omega_\pm^* \Omega_\pm$ is the projection operator with range $\mathfrak{C}$. In the notation of eq. (2.2.9), this statement is equivalent to

$$(k,\Omega_\pm^* \Omega_\pm h) = (k,f) \tag{2.2.12}$$

for every k in $\mathfrak{L}^2$. If $k \perp \mathfrak{C}$, eq. (2.2.12) is obvious. In this case either side vanishes. If $k \in \mathfrak{C}$, we have

$$\left.\begin{aligned} &|(k,\Omega_\pm^* \Omega_\pm h) - (k,f)| = |(\Omega_\pm k, \Omega_\pm f) - (k,f)| \\ &= \lim_{s,t \to \mp\infty} |(\Omega(s)k, \Omega(t)f) - (\Omega(t)k, \Omega(t)f)| \le \lim_{s,t \to \mp\infty} ||[\Omega(s) - \Omega(t)]k||\,||f|| = 0. \end{aligned}\right\} \tag{2.2.13}$$

Hence in this case eq. (2.2.12) is also satisfied. It follows that $\Omega_\pm^* \Omega_\pm$ is the projection onto $\mathfrak{C}$, as we wished to prove. With this result it is easily seen that the operators $\Omega_\pm \Omega_\pm^*$ are likewise projections. Their ranges are $\mathfrak{R}_\pm$. Summarizing, we have

$$\Omega_\pm^* \Omega_\pm = P(\mathfrak{C}), \tag{2.2.14}$$

$$\Omega_\pm \Omega_\pm^* = P(\mathfrak{R}_\pm), \tag{2.2.15}$$

where $P(\mathfrak{C})$ and $P(\mathfrak{R}_\pm)$ denote the projections onto $\mathfrak{C}$ and $\mathfrak{R}_\pm$, respectively.

2.2.3. The intertwining property

If f belongs to $\mathfrak{C}$, we have

$$\lim_{s \to \mp\infty} ||e^{iHt} f_\pm - e^{iHs} e^{-iH_0 s} e^{iH_0 t} f|| = \lim_{s \to \mp\infty} ||e^{iHt} f_\pm - e^{iHt} e^{iH(s-t)} e^{-iH_0(s-t)} f|| = 0. \tag{2.2.16}$$

Hence $\exp(iH_0 t) f$ also belongs to $\mathfrak{C}$, and the operators $\Omega_\pm$ have the intertwining property

$$\Omega_\pm e^{iH_0 t} f = e^{iHt} \Omega_\pm f \qquad (f \in \mathfrak{C}). \tag{2.2.17}$$

Conversely, if $\exp(iH_0 t) f$ belongs to $\mathfrak{C}$, then so does f. This means that if $f \perp \mathfrak{C}$, we also have $\exp(iH_0 t) f \perp \mathfrak{C}$. In the latter case either side of eq. (2.2.17) vanishes, so that this equation is again satisfied. Since every f in $\mathfrak{L}^2$ can be decomposed into a component in $\mathfrak{C}$ and a component orthogonal to $\mathfrak{C}$, it follows that eq. (2.2.17) holds in fact for every f in $\mathfrak{L}^2$. In other words,

$$\Omega_\pm e^{iH_0 t} = e^{iHt} \Omega_\pm. \tag{2.2.18}$$

From eq. (2.2.18) it follows that, if the quantity $\Omega_\pm h$ does not vanish, it cannot be an eigenfunction of H. For let us assume the contrary, i. e. let us assume that

$$e^{iHt}\Omega_{\pm}h = e^{i\lambda t}\Omega_{\pm}h \tag{2.2.19}$$

for every t and some real λ. Then

$$\Omega_{\pm}e^{iH_0t}h = \Omega_{\pm}e^{i\lambda t}h. \tag{2.2.20}$$

Since $\Omega_{\pm}h \neq 0$ by assumption, h must have a component f in $\mathfrak{C}$. Applying $\Omega_{\pm}^{*}$ to both sides of eq. (2.2.20), we obtain

$$e^{iH_0t}f = e^{i\lambda t}f. \tag{2.2.21}$$

From this it follows that

$$\lim_{t\to 0}\left|\left|\frac{1}{t}(e^{iH_0t}-1)f\right|\right| = \lim_{t\to 0}\left|\left|\frac{1}{t}(e^{i\lambda t}-1)f\right|\right| = ||\lambda f||. \tag{2.2.22}$$

Next it follows from the existence of the limit in eq. (2.2.22) that f belongs to the domain of H_0 (Riesz and Sz.-Nagy(11) section 137). Also,

$$\underset{t\to 0}{\mathrm{l.i.m.}}\frac{1}{it}(e^{iH_0t}-1)f = H_0f = \lambda f, \tag{2.2.23}$$

so that f is an eigenfunction of H_0. But since H_0 is known not to have eigenfunctions in $\mathfrak{L}^2$, this result is invalid. Hence the assumption that $\Omega_{\pm}h$ is an eigenfunction of H must be incorrect.

It follows from eq. (2.2.18) that

$$\Omega_{\pm}R_0(\lambda) = R(\lambda)\Omega_{\pm} \qquad (\mathrm{Im}\lambda \neq 0), \tag{2.2.24}$$

where R and R_0 stand for the resolvents of H and H_0, respectively. One method of proving this relation makes use of the formula

$$(g, R(\lambda)f) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(g, e^{iHt}f)dt\int_{-\infty}^{\infty}\frac{1}{l-\lambda}e^{-ilt}dl \tag{2.2.25}$$

given by Stone(12). From eq. (2.2.24) it follows immediately that

$$E_0(l;\Omega_{\pm}^{*}g,f) = E(l;g,\Omega_{\pm}f) \tag{2.2.26}$$

for every f and g in $\mathfrak{L}^2$. Here E and E_0 are the spectral functions associated with H and H_0 which were defined in section 1.4.1. If in eq. (2.2.26) we choose f in $\mathfrak{C}$ and take $g = \Omega_{\pm}f$, we obtain

$$E_0(l;f,f) = E(l;\Omega_{\pm}f,\Omega_{\pm}f). \tag{2.2.27}$$

Now since the spectrum of H_0 is continuous, the left-hand side of eq. (2.2.27) is a continuous function of l. Hence so is the right-hand side. This shows again that $\Omega_{\pm}f$ cannot be an eigenfunction of H. As a matter of fact, if H has eigenfunctions in $\mathfrak{L}^2$, eq. (2.2.27) implies that $\Omega_{\pm} f$ is orthogonal to these (cf. STONE(13) theorem 5.13).

If, given a certain l, there is a function f in $\mathfrak{C}$ such that l is not an interior point of an interval in which $E_0(l;f,f)$ = const., then l certainly belongs to the continuous spectrum of H_0. By eq. (2.2.27), l also belongs to the continuous spectrum of H. Hence the continuous spectrum of H contains the set of points l for which there is a function f with the property mentioned. In practical cases, this set consists of all points l in the spectrum of H_0. The continuous spectrum of H then contains the spectrum of H_0.

For future reference we note that, according to eq. (2.2.26),

$$\Omega_{\pm} E_0(l) = E(l)\Omega_{\pm}, \tag{2.2.28}$$

$E(l)$ and $E_0(l)$ denoting the resolutions of the identity associated with H and H_0, respectively (cf. eq. (1.4.21)). If f belongs to $\mathfrak{C}$, it follows with eq. (1.4.22) that

$$\left.\begin{aligned} ||\Omega_{\pm} E_0(l) f||^2 &= ||E(l)\Omega_{\pm} f||^2 = (\Omega_{\pm} f, E(l)\Omega_{\pm} f) \\ &= (\Omega_{\pm}^* \Omega_{\pm} f, E_0(l) f) = (f, E_0(l) f) = ||E_0(l) f||^2. \end{aligned}\right\} \tag{2.2.29}$$

Hence, if f belongs to $\mathfrak{C}$, so does $E_0(l)f$, by eq. (2.2.14). Now $E_0(l)f$ belongs to $\mathfrak{D}(H_0)$ (cf. section 1.4.2 and ACHIESER and GLASMANN(14) section 66). If l tends to ∞, the function $E_0(l)f$ tends in mean to f. This means that, if f belongs to $\mathfrak{C}$, it can be approximated in mean square by a function which belongs both to $\mathfrak{C}$ and to $\mathfrak{D}(H_0)$.

If f belongs both to $\mathfrak{C}$ and to $\mathfrak{D}(H_0)$, it follows from eqs. (2.2.26) and (1.4.24) that

$$\Omega_{\pm} H_0 f = H\Omega_{\pm} f. \tag{2.2.30}$$

With an argument as used in eq. (2.2.29), it is easily shown that in this case $H_0 f$ also belongs to $\mathfrak{C}$.

2.2.4. The scattering operator

From the beginning of section 2.2.1 we recall that at time t the wave-function in the Schrödinger representation is of the form

$$e^{-iHt} f_+ = e^{-iHt}\Omega_+ f \qquad (f \in \mathfrak{C}). \tag{2.2.31}$$

Now we imagine that in the remote future the system we are considering will behave asymptotically as a free system. Let us therefore determine the probability of finding it in the state $\exp(-iH_0 t)g$, where g belongs to $\mathfrak{C}$. This probability is equal to

$$|(e^{-iH_0 t} g, e^{-iHt}\Omega_+ f)|^2. \tag{2.2.32}$$

If t tends to ∞, this quantity tends to

$$\lim_{t\to\infty} |(e^{iHt} e^{-iH_0 t} g, \Omega_+ f)|^2 = |(\Omega_- g, \Omega_+ f)|^2 = |(g, \Omega_-^* \Omega_+ f)|^2. \tag{2.2.33}$$

Hence the probability for a transition from the state $\exp(-iH_0t)f$ at $t = -\infty$ to the state $\exp(-iH_0t)g$ at $t = \infty$ is determined by the operator

$$S = \Omega_-^*\Omega_+, \tag{2.2.34}$$

which is called the scattering operator.

Since Ω_+f belongs to $\mathfrak{R}_+$, and Ω_-g to $\mathfrak{R}_-$, it is obvious that there are no transitions from f to g if $\mathfrak{R}_+$ and $\mathfrak{R}_-$ are orthogonal. More generally, let us imagine that there exists a function h_+ in $\mathfrak{R}_+$ which is orthogonal to $\mathfrak{R}_-$. By the definition of $\mathfrak{R}_+$, there must be a function h in $\mathfrak{C}$ such that $h_+ = \Omega_+h$. This function has the property that (Ω_-g, Ω_+h) vanishes for every g in $\mathfrak{L}^2$. Hence, if the wave-function takes the form $\exp(-iH_0t)h$ at $t = -\infty$, a situation arises in which there is no possibility for the system to become free as t approaches ∞. Conversely, if there is a function h_- in $\mathfrak{R}_-$ and orthogonal to $\mathfrak{R}_+$, there is a function h in $\mathfrak{C}$ such that the wave-function $\exp(-iH_0t)h$ cannot occur as the outcome of a scattering process. Since either case seems to be pathological, it was required by JAUCH (2) that

$$\mathfrak{R}_+ = \mathfrak{R}_-. \tag{2.2.35}$$

2.2.5. Unitarity

The relation (2.2.35) is directly related to the unitarity of the S-operator. As a matter of fact, if $\mathfrak{R}_+ = \mathfrak{R}_-$, it follows from eqs. (2.2.14), (2.2.15), and (2.2.34) that

$$\left.\begin{aligned} S^*S &= \Omega_+^*\Omega_-\Omega_-^*\Omega_+ = \Omega_+^*P(\mathfrak{R}_-)\Omega_+ = \Omega_+^*P(\mathfrak{R}_+)\Omega_+ = \Omega_+^*\Omega_+ = P(\mathfrak{C}),\\ SS^* &= \Omega_-^*\Omega_+\Omega_+^*\Omega_- = \Omega_-^*P(\mathfrak{R}_+)\Omega_- = \Omega_-^*P(\mathfrak{R}_-)\Omega_- = \Omega_-^*\Omega_- = P(\mathfrak{C}).\end{aligned}\right\} \tag{2.2.36}$$

Hence S can be considered as a unitary operator in $\mathfrak{C}$. In all cases which were investigated explicitly thus far, $\mathfrak{C}$ was the whole space $\mathfrak{L}^2$. Hence S was unitary whenever $\mathfrak{R}_+ = \mathfrak{R}_-$.

It is easily seen from eq. (2.2.36) that for S to be unitary it is not merely sufficient, but it is also necessary that $\mathfrak{R}_+ = \mathfrak{R}_-$. Thus special studies were devoted to this problem by KURODA (6, 15) and IKEBE (16). For the case of two particles, hence three-dimensional $\boldsymbol{x}$, it was shown by KURODA (6, 15) that S is unitary if the interaction $V(\boldsymbol{x})$ is both integrable and square-integrable. If $V(\boldsymbol{x})$ is spherically symmetric, it is sufficient if there is a positive ζ such that $V(|\boldsymbol{x}|)(1+|\boldsymbol{x}|)^{-\frac{1}{2}+\zeta}$ is square-integrable (KURODA (6)). IKEBE (16) established the unitarity of the S-operator under the assumption that $V(\boldsymbol{x})$ is Hölder-continuous except for a finite number of singularities, is square-integrable, and as $|\boldsymbol{x}|$ tends to ∞ is of the order $O(|\boldsymbol{x}|^{-2-\eta})$, with some positive η. Under these assumptions he was also able to show that the space $\mathfrak{R}_+ = \mathfrak{R}_-$ is the orthogonal complement of the space spanned by the eigenfunctions of H, if these exist. Hence, under Ikebe's assumptions on the interaction, every function which is orthogonal to the eigenfunctions of H can occur as the wave-function in a scattering process. In section 2.2.3 we saw already that in a scattering experiment we never get the eigenfunctions of H. This follows from the result that $\Omega_{\pm}f$ is orthogonal to these eigenfunctions (cf. eq. (2.2.27)).

2.2.6. Integral representations of the wave-operators

For future reference we note that it was shown by Jauch (2) that, if the limits (2.2.4) and (2.2.5) exist,

$$\left.\begin{aligned} f_{+} &= \underset{\varepsilon\to 0}{\text{l.i.m.}}\, \varepsilon \int_{-\infty}^{0} e^{\varepsilon t}\Omega(t)dtf, \\ f_{-} &= \underset{\varepsilon\to 0}{\text{l.i.m.}}\, \varepsilon \int_{0}^{\infty} e^{-\varepsilon t}\Omega(t)dtf, \end{aligned}\right\} \quad (2.2.37)$$

it being understood that ε tends to 0 through positive values.

Since for fixed f and g the quantity $(g,\Omega(t)f)$ is a bounded and continuous function of t, the integrals

$$\mp \varepsilon \int_{0}^{\mp\infty} e^{-\varepsilon|t|}(g,\Omega(t)f)dt \quad (2.2.38)$$

exist and are bounded. Hence, in virtue of the Riesz-Fréchet theorem (Achieser and Glasmann (14) section 21), there are bounded operators $\Omega_{\pm\varepsilon}$, which conveniently may be written as

$$\Omega_{\pm\varepsilon} = \mp \varepsilon \int_{0}^{\mp\infty} e^{-\varepsilon|t|}\Omega(t)dt, \quad (2.2.39)$$

such that

$$\mp \varepsilon \int_{0}^{\mp\infty} e^{-\varepsilon|t|}(g,\Omega(t)f)dt = \mp \varepsilon\left(g, \int_{0}^{\mp\infty} e^{-\varepsilon|t|}\Omega(t)dtf\right). \quad (2.2.40)$$

This defines the integrals in eq. (2.2.37).

Now if, given a positive δ, there exists a number T such that

$$||f_{+} - \Omega(t)f|| < \delta \quad (2.2.41)$$

whenever $t < -T$, it follows that

$$\left.\begin{aligned} &||f_{+} - \varepsilon \int_{-\infty}^{0} e^{\varepsilon t}\Omega(t)dtf|| = \varepsilon|| \int_{-\infty}^{0} e^{\varepsilon t}dt[f_{+} - \Omega(t)f]|| \\ &< \varepsilon \int_{-\infty}^{-T} e^{\varepsilon t}||f_{+} - \Omega(t)f||dt + \varepsilon \int_{-T}^{0} e^{\varepsilon t}||f_{+} - \Omega(t)f||dt < \delta + 2(1 - e^{-\varepsilon T})||f||. \end{aligned}\right\} \quad (2.2.42)$$

By choosing ε sufficiently small, this can be made less than 2δ, say. The first statement of eq. (2.2.37) then follows. The second one can be justified similarly.

2.3. Multi-channel scattering

2.3.1. The channel concept

To discuss the scattering in a system of more than two particles, we consider the case that in the distant past the system was split into m fragments which were very far apart. In this situation it is convenient to introduce m' sets of internal coordinates to describe the motion within the m' fragments that consist of two or more particles, $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{m'} (m' \leq m)$, plus a set of $m-1$ three-dimensional coordinates, $\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}$, for the motion of the fragments with respect to each other. It follows from section 1.2.1 that this can be done in such a way that the differential operator for the relative kinetic energy of the m fragments with respect to each other takes the form $-\sum_{j=m+1}^{2m-1} \Delta(\boldsymbol{x}_j)$.

To bring out the assumption that in the distant past the m fragments were very far apart, and that effectively there was no interaction between them, we introduce the operator

$$H'_a(\boldsymbol{x}) = -\sum_{j=m+1}^{2m-1} \Delta(\boldsymbol{x}_j) + \lambda_a, \tag{2.3.1}$$

where λ_a is a real number to be determined in the course of the following. In the space $\mathfrak{L}^2(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{m'}, \boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1})$ the operator $H'_a(\boldsymbol{x})$ has a unique self-adjoint extension. This we denote by H_a (cf. the discussion in section 1.7.2). Now the idea is that H_a represents the effective energy of the system at time $t = -\infty$. That is, we imagine that in the distant past the system behaved according to some wave-function $\exp(-iH_a t) f_a$, and we require the existence of

$$f_{a\pm} = \underset{t \to \mp\infty}{\text{l.i.m.}}\, e^{iHt} e^{-iH_a t} f_a. \tag{2.3.2}$$

In this picture, λ_a plays the role of the intrinsic energy of the m fragments. Under very general assumptions on the interaction between the fragments, it is shown in section 2.4.3 that, for the limit in eq. (2.3.2) to exist, it is necessary that λ_a is of the form

$$\lambda_a = \sum_{j=1}^{m'} \lambda_{(j)}, \tag{2.3.3}$$

where the numbers $\lambda_{(j)}$ are eigenvalues of the Hamiltonians $H_{(j)}(\boldsymbol{x}_j)$ for the internal motions of the respective fragments. It is also shown that, if to each $\lambda_{(j)}$ in the series (2.3.3) there corresponds only one eigenfunction $\varphi_{(j)}(\boldsymbol{x}_j)$, the function f_a must be of the form

$$f_a(\boldsymbol{x}) = \prod_{j=1}^{m'} \varphi_{(j)}(\boldsymbol{x}_j) f(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}). \tag{2.3.4}$$

If $\lambda_{(1)}$ is degenerate, with orthonormal eigenfunctions $\varphi_{(1)1}(\boldsymbol{x}_1)$ and $\varphi_{(1)2}(\boldsymbol{x}_1)$ say, we find it convenient to write $H_a = H_b$, and to consider separately

$$\left.\begin{aligned} f_a(\boldsymbol{x}) &= \varphi_{(1)1}(\boldsymbol{x}_1)\prod_{j=2}^{m'}\varphi_{(j)}(\boldsymbol{x}_j)f(\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1}),\\ f_b(\boldsymbol{x}) &= \varphi_{(1)2}(\boldsymbol{x}_1)\prod_{j=2}^{m'}\varphi_{(j)}(\boldsymbol{x}_j)f(\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1}). \end{aligned}\right\} \tag{2.3.5}$$

And similarly for more degenerate cases.

If asymptotically the system behaves according to the wave-function $\exp(-iH_a t)f_a$, it is said to be in channel a. With the notation outlined in the previous paragraph, we have $H_a \neq H_b$, except possibly if $f_a \perp f_b$. The case $H_a = H_b$, $f_a \perp f_b$ also covers the exceptional situation that λ_a can be decomposed into a sum of the form (2.3.3) in more than one way.

2.3.2. The wave-operators

In line with section 2.2.2 we denote the set of functions f_a for which the limits (2.3.2) exist by $\mathfrak{C}_a$, the sets of functions $f_{a\pm}$ by $\mathfrak{R}_{a\pm}$. These sets are all closed. Decomposing a general function h in $\mathfrak{L}^2$ according to

$$h = f_a + g_a, \quad f_a \in \mathfrak{C}_a, \quad g_a \perp \mathfrak{C}_a, \tag{2.3.6}$$

we define the wave-operators $\Omega_{a\pm}$ by

$$\Omega_{a\pm} h = \underset{t\to\mp\infty}{\mathrm{l.i.m.}}\,\Omega_a(t)f_a, \tag{2.3.7}$$

where

$$\Omega_a(t) = e^{iHt}e^{-iH_a t}. \tag{2.3.8}$$

In analogy to eqs. (2.2.14) and (2.2.15), we have

$$\Omega^*_{a\pm}\Omega_{a\pm} = P(\mathfrak{C}_a), \tag{2.3.9}$$

$$\Omega_{a\pm}\Omega^*_{a\pm} = P(\mathfrak{R}_{a\pm}). \tag{2.3.10}$$

Also,

$$\Omega_{a\pm}e^{iH_a t} = e^{iHt}\Omega_{a\pm}. \tag{2.3.11}$$

With trivial changes of notation, the conclusions which were drawn from the corresponding equation (2.2.18) apply also in the present case. In particular, if R_a denotes the resolvent of H_a, there is a relation analogous to eq. (2.2.24). The spectrum of H contains the spectrum of H_a if, given a point l in the spectrum of H_a, there is a function f_a in $\mathfrak{C}_a$ such that l is not an interior point of an interval in which $E_a(l; f_a, f_a) =$ const. If f_a belongs to $\mathfrak{C}_a$, the function $E_a(l)f_a$ belongs both to $\mathfrak{C}_a$ and to $\mathfrak{D}(H_a)$.

If at time $t = -\infty$ the system was in the state $\exp(-iH_a t)f_a$, the probability of finding it in the state $\exp(-iH_b t)g_b$ at time $t = \infty$ is given by

$$\lim_{t\to\infty}|(e^{-iH_b t}g_b, e^{-iHt}\Omega_{a+}f_a)|^2 = |(g_b, \Omega^*_{b-}\Omega_{a+}f_a)|^2. \tag{2.3.12}$$

In evaluating this expression in practical cases, it will be necessary to express H_b and g_b in the coordinate system adapted to H_a and f_a, or vice versa. This may give awkward formulas, but we shall not bother about that in the following.

It follows from eq. (2.3.12) that the transitions from channel a to channel b are determined by the operator

$$S_{ba} = \Omega^*_{b-}\Omega_{a+}. \tag{2.3.13}$$

If $\mathfrak{R}_{a+} \perp \mathfrak{R}_{b-}$, no transitions are possible.

2.3.3. Orthogonality of the channels

In connection with the foregoing, the question arises whether there is an ambiguity in the channel concept in the sense that the statement that the system is in channel b does not exclude its being in channel c. Let us assume for a moment that the system can be in two channels at the same time. Or rather, let us assume that there is a function h which belongs to $\mathfrak{R}_{a+}$, and also to both $\mathfrak{R}_{b-}$ and $\mathfrak{R}_{c-}$. By definition, this assumption means that there are functions h_a, h_b, and h_c such that

$$h = \Omega_{a+}h_a = \Omega_{b-}h_b = \Omega_{c-}h_c. \tag{2.3.14}$$

If we now consider the wave-function $\exp(-iHt)\Omega_{a+}h_a$, we find

$$\left.\begin{aligned} \lim_{t\to\infty} \|e^{-iHt}\Omega_{a+}h_a - e^{-iH_bt}h_b\| &= 0,\\ \lim_{t\to\infty} \|e^{-iHt}\Omega_{a+}h_a - e^{-iH_ct}h_c\| &= 0. \end{aligned}\right\} \tag{2.3.15}$$

According to this result, the wave-function in question would tend to $\exp(-iH_bt)h_b$, and also to $\exp(-iH_ct)h_c$. Hence, at $t = \infty$, the system would be in two different channels. It is clear that the root of this undesirable situation is the idea that there is an overlap between the sets $\mathfrak{R}_{b-}$ and $\mathfrak{R}_{c-}$. It is therefore important that

$$\mathfrak{R}_{a+} \perp \mathfrak{R}_{b+}, \qquad \mathfrak{R}_{a-} \perp \mathfrak{R}_{b-}, \qquad (a \neq b), \tag{2.3.16}$$

so that the above ambiguity does not occur. Obviously we do not expect a relation of the form $\mathfrak{R}_{a+} \perp \mathfrak{R}_{b-}$, because this would exclude the possibility of transitions from one channel to another in the course of time.

Equation (2.3.16) is due to Jauch (3). For the proof it suffices to show that

$$(\Omega_{b\pm}g_b, \Omega_{a\pm}f_a) = 0 \qquad (a \neq b) \tag{2.3.17}$$

for every f_a in $\mathfrak{C}_a$ and every g_b in $\mathfrak{C}_b$. Now,

$$\left.\begin{aligned} &|(\Omega_b(t)g_b, \Omega_a(t)f_a) - (\Omega_{b\pm}g_b, \Omega_{a\pm}f_a)| = |(\Omega_b(t)g_b, [\Omega_a(t) - \Omega_{a\pm}]f_a)\\ &+ ([\Omega_b(t) - \Omega_{b\pm}]g_b, \Omega_{a\pm}f_a)| \leq \|g_b\|\, \|[\Omega_a(t) - \Omega_{a\pm}]f_a\| + \|[\Omega_b(t) - \Omega_{b\pm}]g_b\|\, \|f_a\|. \end{aligned}\right\} \tag{2.3.18}$$

Since the right-hand side of this relation tends to 0 as t tends to $\mp\infty$, it follows that

$$(\Omega_{b\pm}g_b, \Omega_{a\pm}f_a) = \lim_{t\to\mp\infty} (\Omega_b(t)g_b, \Omega_a(t)f_a). \tag{2.3.19}$$

Hence, by the definition of $\Omega(t)$,

$$(\Omega_{b\pm}g_b, \Omega_{a\pm}f) = \lim_{t\to\mp\infty} (e^{-iH_bt}g_b, e^{-iH_at}f_a) = \lim_{t\to\mp\infty} (g_b, e^{-i(H_a-H_b)t}f_a), \tag{2.3.20}$$

where the third member is justified by the fact that H_a and H_b commute.

We must now distinguish two cases. Firstly the case that $H_a = H_b$. Then the right-hand side of eq. (2.3.20) vanishes because f_a and g_b are orthogonal (cf. the discussion at the end of section 2.3.1). Hence eq. (2.3.17) is satisfied. Secondly we consider the case that $H_a \neq H_b$. With the definition $H_a - H_b = K$, it is then a question of

$$\lim_{t\to\mp\infty} (g_b, e^{-iKt}f_a). \tag{2.3.21}$$

Now the quantity $(g_b,\exp(-iKt)f_a)$ is bounded, and it is known to tend to a limit as t tends to $\mp\infty$, by eq. (2.3.20). From this it follows that we must have

$$\lim_{t\to\mp\infty} (g_b, e^{-iKt}f_a) = \lim_{t\to\mp\infty} \frac{1}{t}\int_0^t (g_b, e^{-iKs}f_a)ds. \tag{2.3.22}$$

But, according to one of the ergodic theorems of VON NEUMANN (17), the right-hand side of eq. (2.3.22) is equal to (g_b,Pf_a), where P is the projection operator onto the subspace of $\mathfrak{L}^2$ spanned by the functions h with $\exp(-iKt)h = h$. These functions are eigenfunctions of K satisfying $Kh = 0$ (cf. the argument following eq. (2.2.21)). However, it is not difficult to see that K has no eigenfunctions in $\mathfrak{L}^2$. Hence P is in fact the zero operator. Thus the expression (2.3.21) vanishes, and eq. (2.3.17) is again satisfied. Taking into account that $\Omega^*_{b\pm}\Omega_{a\pm}f_a$ belongs to $\mathfrak{C}_b$, by eq. (2.2.11), it follows that

$$\Omega^*_{b\pm}\Omega_{a\pm}f_a = \delta_{ba}f_a. \tag{2.3.23}$$

This settles the orthogonality of the channels. It is not claimed that $\mathfrak{C}_a \perp \mathfrak{C}_b$ if $a \neq b$. As a matter of fact, a relation of this sort does in general not hold true, nor is there a reason why we should want it to be satisfied. Indeed, the functions f_a and g_b are only auxiliary quantities. What counts is the wave-function, which is of the form $\exp(-iHt)\Omega_{a+}f_a$. The crucial point is that this can be decomposed unambiguously into mutually orthogonal components corresponding to the various channels. The decomposition is brought about by the projection operators $\Omega_{b-}\Omega^*_{b-}$, which project onto the sets $\mathfrak{R}_{b-}$. The component corresponding to channel b is given by

$$\Omega_{b-}\Omega^*_{b-}e^{-iHt}\Omega_{a+}f_a = e^{-iHt}\Omega_{b-}\Omega^*_{b-}\Omega_{a+}f_a. \tag{2.3.24}$$

In fact, owing to the relation

$$\lim_{t\to\infty} ||e^{-iH_bt}\Omega_{b-}^*\Omega_{a+}f_a - e^{-iHt}\Omega_{b-}\Omega_{b-}^*\Omega_{a+}f_a|| = 0, \tag{2.3.25}$$

either side of eq. (2.3.24) tends to $\exp(-iH_bt)\Omega_{b-}^*\Omega_{a+}f_a$ as t tends to ∞. Otherwise stated, the component of $\exp(-iHt)\Omega_{a+}f_a$ which at $t = \infty$ will be in channel b is its projection onto $\mathfrak{R}_{b-}$. It follows from eq. (2.3.11) that the projection operator $\Omega_{b-}\Omega_{b-}^*$ commutes with $\exp(-iHt)$. This means that the decomposition according to channels does not depend on the time at which it is made. Also, it is unambiguous, in virtue of the orthogonality of the sets $\mathfrak{R}_{b-}$.

It is often convenient to write the wave-function $\exp(-iHt)\Omega_{a+}f_a$ as the sum of an incident wave plus a scattered wave, the latter taking the form

$$e^{-iHt}\Omega_{a+}f_a - e^{-iH_at}f_a. \tag{2.3.26}$$

If the scattered wave is decomposed according to channels, it follows with eq. (2.3.11) that the probability of scattering into channel b is given by

$$||\Omega_{b-}^*(e^{-iHt}\Omega_{a+}f_a - e^{-iH_at}f_a)||^2 = ||\Omega_{b-}^*(\Omega_{a+} - e^{iHt}e^{-iH_at})f_a||^2. \tag{2.3.27}$$

If t tends to ∞, this tends to

$$||\Omega_{b-}^*(\Omega_{a+} - \Omega_{a-})f_a||^2 = ||(S_{ba} - \delta_{ba})f_a||^2. \tag{2.3.28}$$

2.3.4. Completeness of the channel description

Since the space $\mathfrak{L}^2$ is separable and the closed sets $\mathfrak{R}_{b\pm}$ are mutually orthogonal, the number of channels is finite or denumerably infinite. In the former case it is obvious that the operators $\sum_b\Omega_{b\pm}\Omega_{b\pm}^*$ are projections. In the latter case it can be shown that, if N tends to ∞, the sequences of projections $\sum_{b=1}^N\Omega_{b\pm}\Omega_{b\pm}^*$ tend to operators which are again projections. Indeed, according to Stone's (13) theorem 2.40, there exist projections $P_\pm$, which may be denoted by $\sum_b\Omega_{b\pm}\Omega_{b\pm}^*$, such that

$$\lim_{N\to\infty} \left|\left|\left[\sum_{b=1}^N \Omega_{b\pm}\Omega_{b\pm}^* - \sum_b \Omega_{b\pm}\Omega_{b\pm}^*\right]f\right|\right| = 0 \tag{2.3.29}$$

for every f in $\mathfrak{L}^2$. From this result it is obvious that

$$(g, \sum_b \Omega_{b\pm}\Omega_{b\pm}^* f) = \sum_b (g, \Omega_{b\pm}\Omega_{b\pm}^* f). \tag{2.3.30}$$

The ranges of the two limit-projections are the sets $\mathfrak{R}_\pm = \sum_b \oplus \mathfrak{R}_{b\pm}$, that is the closed sets determined by the sums $\mathfrak{R}_{1\pm} + \mathfrak{R}_{2\pm} + \ldots$. In other words,

$$\sum_b \Omega_{b\pm}\Omega_{b\pm}^* = P(\mathfrak{R}_\pm). \tag{2.3.31}$$

Now we expect on physical grounds that, if b runs through all channels,

$$\sum_b \Omega_{b-}\Omega^*_{b-}\Omega_{a+}f_a = \Omega_{a+}f_a, \tag{2.3.32}$$

so that $\Omega_{a+}f_a$ does not have components which do not correspond to some channel b. Since $\Omega_{a+}f_a$ runs through $\mathfrak{R}_{a+}$ when f_a runs through $\mathfrak{C}_a$, it is clear that, for eq. (2.3.32) to be satisfied for every f_a in $\mathfrak{C}_a$, it is necessary that

$$\mathfrak{R}_{a+} \subseteq \mathfrak{R}_-. \tag{2.3.33}$$

On the other hand, in order that every function of the form $\exp(-iH_a t)f_a$ can be realized as the result of a scattering event, we must have

$$\mathfrak{R}_+ \supseteq \mathfrak{R}_{a-}. \tag{2.3.34}$$

Hence, combining eqs. (2.3.33) and (2.3.34), we expect that

$$\mathfrak{R}_+ = \mathfrak{R}_-. \tag{2.3.35}$$

This is a generalization of eq. (2.2.35).

It is not known at present if this relation holds true in practical cases. There is no reason to believe that it does not. But only in one fairly special multi-channel example has it been possible to check it (18, 19). For the general case of a system of n particles with two-body interactions depending only on the distances between the particles, no methods for investigating this problem seem to be known.

2.3.5. Unitarity

In the multi-channel case there is a connection between the relation (2.3.35) and unitarity in the following sense. Let $\mathfrak{H}$ be the set of all column-matrices $f = \{f_a\}$ which have a function $f_a \in \mathfrak{C}_a$ in row a, with $\sum_a ||f_a||^2 < \infty$. If the inner product is defined by

$$(g, f) = \sum_a (g_a, f_a), \tag{2.3.36}$$

and if addition and multiplication by a constant are defined in the natural way, $\mathfrak{H}$ is a Hilbert space. We now consider the operator-matrix $\mathcal{S}$ which transforms $\{f_a\}$ according to

$$\mathcal{S}\{f_a\} = \{\sum_b S_{ab} f_b\} = \{\sum_b \Omega^*_{a-}\Omega_{b+} f_b\}. \tag{2.3.37}$$

In case the number of channels is infinite, there is a convergence problem involved in this equation. However, in virtue of eqs. (2.3.17) and (2.3.9),

$$||\sum_{b=M}^{N} \Omega_{b+}f_b||^2 = \sum_{b=M}^{N} ||\Omega_{b+}f_b||^2 = \sum_{b=M}^{N} ||f_b||^2. \tag{2.3.38}$$

If M and N tend to ∞, this tends to 0, by the assumption that $\sum_a \|f_a\|^2$ converges. Hence the quantity $\sum_{b=1}^{N} \Omega_{b+} f_b$ tends in mean to a function in $\mathfrak{L}^2$, which we denote by $\sum_b \Omega_{b+} f_b$. Applying the operator Ω^*_{a-} to this function, we obtain a function in $\mathfrak{C}_a$ (cf. the discussion following eq. (2.2.11)). Also, since Ω^*_{a-} is bounded and therefore continuous,

$$\Omega^*_{a-} \sum_b \Omega_{b+} f_b = \underset{N\to\infty}{\text{l.i.m.}} \sum_{b=1}^{N} \Omega^*_{a-} \Omega_{b+} f_b = \sum_b \Omega^*_{a-} \Omega_{b+} f_b. \tag{2.3.39}$$

This defines the third member of eq. (2.3.37) for the case of an infinite number of channels. It follows that in this expression the function in row a belongs to $\mathfrak{C}_a$.

As regards the norm of $\mathcal{S}f$ we have

$$\left.\begin{aligned} (\mathcal{S}f, \mathcal{S}f) &= \sum_b \Big(\Omega^*_{b-} \sum_c \Omega_{c+} f_c, \Omega^*_{b-} \sum_a \Omega_{a+} f_a \Big) \\ &= \Big(\sum_c \Omega_{c+} f_c, P(\mathfrak{R}_-) \sum_a \Omega_{a+} f_a \Big) \le \|\sum_a \Omega_{a+} f_a\|^2 = \sum_a \|f_a\|^2 < \infty . \end{aligned}\right\} \tag{2.3.40}$$

Hence $\mathcal{S}$ is a bounded operator in $\mathfrak{H}$.

In the above argument the sign of equality applies if and only if the relation (2.3.33) holds true for every a. In this case we have

$$(\mathcal{S}g, \mathcal{S}f) = \Big(\sum_c \Omega_{c+} g_c, \sum_a \Omega_{a+} f_a \Big) = \sum_a (g_a, f_a) = (g, f), \tag{2.3.41}$$

which is equivalent to $\mathcal{S}^* \mathcal{S} = \mathcal{I}$, $\mathcal{I}$ denoting the unit matrix.

The adjoint operator $\mathcal{S}^*$ satisfies

$$(\mathcal{S}^* g, f) = (g, \mathcal{S}f) = \sum_a \Big(g_a, \sum_b \Omega^*_{a-} \Omega_{b+} f_b \Big). \tag{2.3.42}$$

This can easily be transformed into

$$(\mathcal{S}^* g, f) = \sum_b \Big(\Omega^*_{b+} \sum_a \Omega_{a-} g_a, f_b \Big), \tag{2.3.43}$$

from which it follows that

$$(\mathcal{S}^* g, \mathcal{S}^* f) = \sum_b \Big(\Omega^*_{b+} \sum_a \Omega_{a-} g_a, \Omega^*_{b+} \sum_c \Omega_{c-} f_c \Big). \tag{2.3.44}$$

Hence we obtain

$$(\mathcal{S}^* g, \mathcal{S}^* f) = \Big(\sum_a \Omega_{a-} g_a, \sum_c \Omega_{c-} f_c \Big) = \sum_a (g_a, f_a) = (g, f) \tag{2.3.45}$$

if and only if eq. (2.3.34) holds true for every a. Summarizing, the unitarity relations $\mathcal{S}^* \mathcal{S} = \mathcal{I}$ and $\mathcal{S}\mathcal{S}^* = \mathcal{I}$ are satisfied if and only if eq. (2.3.35) is fulfilled.

To avoid confusion, we remark that the matrix $\mathcal{S}$ does not occur in Jauch's paper (3). In particular, it is not the quantity Jauch denotes by S. Although there is a certain analogy between the present $\mathcal{S}$-matrix and the operator S discussed in sections 2.2.4 and 2.2.5, it must be observed that there does not seem to be a sensible way of writing $\mathcal{S}$ as the product of a matrix Ω_-^* times a matrix Ω_+ in such a way that the Ω-matrices are limits of suitable time-dependent matrices which involve Hamiltonians.

2.3.6. Conjugation and symmetry

For the systems in which we are interested in the present paper, the relations (2.3.33) and (2.3.34) are not independent. This is due to the fact that there is an operator of conjugation C,

$$(Cg, Cf) = (f, g), \qquad C^2 f = f, \tag{2.3.46}$$

such that

$$CHf = HCf \qquad CH_a f = H_a Cf \tag{2.3.47}$$

for every f in $\mathfrak{D}(H)$, or every f in $\mathfrak{D}(H_a)$, as the case may be. Indeed, if C is defined by the relation $Cf(\boldsymbol{x}) = \bar{f}(\boldsymbol{x})$ for every f in $\mathfrak{L}^2$, then eqs. (2.3.46) and (2.3.47) are satisfied.

It follows from eq. (2.3.47) that

$$R(\lambda) = CR(\bar{\lambda})C \tag{2.3.48}$$

(Achieser and Glasmann (14) section 45). Hence, if f and g are any two functions in $\mathfrak{L}^2$, eq. (1.4.28) yields

$$\left.\begin{aligned}(g, Ce^{iHt}f) = (e^{iHt}f, Cg) &= -\frac{1}{2\pi i}\lim_{U_\pm \to \pm\infty}\lim_{\zeta\to 0}\int_{U_-}^{U_+} e^{-iut}([R(u+i\zeta) - R(u-i\zeta)]f, Cg)du \\ &= \frac{1}{2\pi i}\lim_{U_\pm \to \pm\infty}\lim_{\zeta\to 0}\int_{U_-}^{U_+} e^{-iut}(g, [R(u+i\zeta) - R(u-i\zeta)]Cf)du = (g, e^{-iHt}Cf).\end{aligned}\right\} \tag{2.3.49}$$

Since in this relation f and g are arbitrary,

$$Ce^{iHt} = e^{-iHt}C. \tag{2.3.50}$$

Now let g be any function in $\mathfrak{R}_{a+}$. Then there is a function f_a in $\mathfrak{C}_a$ such that

$$\lim_{t\to-\infty} \|g - e^{iHt}e^{-iH_a t}f_a\| = 0. \tag{2.3.51}$$

But from this it follows that

$$\lim_{t\to\infty} \|Cg - e^{iHt}e^{-iH_a t}Cf_a\| = 0. \tag{2.3.52}$$

Hence, if g belongs to $\mathfrak{R}_{a+}$, then Cg belongs to $\mathfrak{R}_-$. More generally, if g belongs to $\mathfrak{R}_+$, then Cg belongs to $\mathfrak{R}_-$, and vice versa.

Let us now assume that eq. (2.3.33) holds true for every a. Then, if g belongs to $\mathfrak{R}_+$, it also belongs to $\mathfrak{R}_-$. We know already that, if h is any function in $\mathfrak{R}_-$, the function Ch belongs to $\mathfrak{R}_+$. If eq. (2.3.33) is satisfied, it follows that Ch also belongs to $\mathfrak{R}_-$. Hence h belongs to $\mathfrak{R}_+$. But this means that $\mathfrak{R}_- \subseteq \mathfrak{R}_+$, i. e. that eq. (2.3.34) holds true. We thus see that in problems which admit a conjugation, eq. (2.3.34) is a consequence of eq. (2.3.33). Conversely, eq. (2.3.33) is a consequence of eq. (2.3.34).

If channel a is not degenerate, it follows unambiguously from eq. (2.3.52) that

$$C\Omega_{a+}f_a = \Omega_{a-}Cf_a. \tag{2.3.53}$$

If there are degenerate channels $a,b,\ldots$ with $H_a = H_b = \ldots$, the function Cf_a does not necessarily belong to $\mathfrak{C}_a$. However, by an appropriate choice of the channels, there will in many cases be pairs of channels a,a', with $H_a = H_{a'}$, such that the conjugation C transforms a function f_a in $\mathfrak{C}_a$ into a function in $\mathfrak{C}_{a'}$, a function $f_{a'}$ in $\mathfrak{C}_{a'}$ into a function in $\mathfrak{C}_a$. If this is so, we have

$$C\Omega_{a\pm}f_a = \Omega_{a'\mp}Cf_a, \qquad C\Omega_{a'\pm}f_{a'} = \Omega_{a\mp}Cf_{a'}. \tag{2.3.54}$$

In the matrix notation of the previous section we now define

$$\mathcal{C}f = \mathcal{C}\{f_a\} = \{Cf_{a'}\}. \tag{2.3.55}$$

After some rearrangements this yields

$$\left.\begin{aligned}(g, \mathcal{C}\mathcal{S}^*\mathcal{C}f) &= (\mathcal{C}f, \mathcal{S}\mathcal{C}g) = \sum_{a,b}(Cf_{a'}, \Omega_{a-}^*\Omega_{b+}Cg_{b'})\\ &= \sum_{a,b}(C\Omega_{b+}Cg_{b'}, C\Omega_{a-}Cf_{a'}) = \sum_{a,b}(g_{b'}, \Omega_{b'-}^*\Omega_{a'+}f_{a'}) = (g, \mathcal{S}f).\end{aligned}\right\} \tag{2.3.56}$$

It is appropriate to call $\mathcal{C}\mathcal{S}^*\mathcal{C}$ the transpose of $\mathcal{S}$. In this sense we can say that the transpose of $\mathcal{S}$ is equal to $\mathcal{S}$, hence that $\mathcal{S}$ is symmetric with respect to the conjugation $\mathcal{C}$.

2.3.7. The optical theorem

For future reference we want to say a few words about the optical theorem and its connection with the unitarity of the S-operator (cf. Messiah (20) ch. XIX, section 31). In its simplest form, the optical theorem gives a relation between the total intensity scattered from any particular channel, and the amplitude for scattering into the channel itself. In the one-channel case, one usually argues as follows.

In the Schrödinger representation the scattered wave has the form

$$e^{-iHt}\Omega_+f - e^{-iH_0t}f. \tag{2.3.57}$$

In the interaction representation it is

$$e^{iH_0t}e^{-iHt}\Omega_+f - f. \tag{2.3.58}$$

As t tends to ∞, this is supposed to tend to

$$\underset{t\to\infty}{\text{l.i.m.}}\, e^{iH_0t}e^{-iHt}\Omega_+f - f = \Omega_-^*\Omega_+f - f = Sf - f, \tag{2.3.59}$$

a relation which in fact only holds true if $\mathfrak{R}_- \supseteq \mathfrak{R}_+$. Now if $S^*S = 1$,

$$||[S-1]f||^2 = ||Sf||^2 + ||f||^2 - (Sf,f) - (f,Sf) = 2\mathrm{Re}(f,[1-S]f). \tag{2.3.60}$$

Hence one expects that, as t tends to ∞, the intensity of the scattered wave tends to the right-hand side of eq. (2.3.60). If this is confirmed by experiments, it might be taken to indicate that $S^*S = 1$, hence that $\mathfrak{R}_- \supseteq \mathfrak{R}_+$. But such a conclusion would be premature. For what one observes experimentally is the intensity at a very large but finite time t. This is

$$||e^{-iHt}\Omega_+f - e^{-iH_0t}f||^2 = ||\Omega_+f - e^{iHt}e^{-iH_0t}f||^2. \tag{2.3.61}$$

If t tends to ∞, the intensity tends to

$$||\Omega_+f - \Omega_-f||^2 = 2||f||^2 - (\Omega_+f,\Omega_-f) - (\Omega_-f,\Omega_+f) = 2\mathrm{Re}(f,[1-S]f). \tag{2.3.62}$$

Hence there is a limit of the form (2.3.62) irrespective as to whether S is unitary.

In obtaining this result, the crucial point is that the intensity is determined first, the limit next. This corresponds to the experimental situation. In a multi-channel problem, it is in general not even possible to determine the limit first. For the quantity

$$e^{iH_at}e^{-iHt}\Omega_{a+}f_a - f_a \tag{2.3.63}$$

does in general not tend to a limit. Hence there is no multi-channel analogue to eq. (2.3.59). At the same time, the intensity of the wave scattered from channel a is given by

$$||e^{-iHt}\Omega_{a+}f_a - e^{-iH_at}f_a||^2 = ||\Omega_{a+}f_a - e^{iHt}e^{-iH_at}f_a||^2. \tag{2.3.64}$$

If t tends to ∞, this tends to

$$||\Omega_{a+}f_a - \Omega_{a-}f_a||^2 = 2\mathrm{Re}(f_a,[1-S_{aa}]f_a), \tag{2.3.65}$$

analogously to eq. (2.3.62).

With the methods of section 2.3.4, it is easily checked that

$$\sum_b ||(S_{ba}-\delta_{ba})f_a||^2 = 2\mathrm{Re}(f_a,[1-S_{aa}]f_a) \tag{2.3.66}$$

if and only if $\mathfrak{R}_- \supseteq \mathfrak{R}_{a+}$. Hence it is only under this condition that the total scattering intensity is the sum of the intensities scattered into the separate channels. If eq. (2.3.66) holds true for every a, we have

$$||[\mathscr{S} - \mathscr{I}]f||^2 = 2\mathrm{Re}\big(f,[\mathscr{I} - \mathscr{S}]f\big). \tag{2.3.67}$$

This is the multi-channel analogue of eq. (2.3.60). It is the point of the present section that in this expression it is the right-hand side which is simply related to the total scattering intensity, rather than the left-hand side, as one might be inclined to think.

It is shown in section 2.8.9 how in special cases the intensity of the scattered wave is connected with the forward scattering amplitude.

2.4. The existence of the wave-operators

2.4.1. A general condition

For the remaining part of the present investigation, we restrict ourselves explicitly to the class of n-particle systems described in section 2.1.2. For this class we first find sufficient conditions on the interaction under which there exist wave-operators. The method for obtaining such conditions is mainly due to JAUCH and ZINNES (5). However, because we have restricted ourselves from the outset to interactions satisfying eq. (2.1.5), we need only a simplified version of the argument of these authors.

Let us observe first of all that, if we want to show that there is a limit of the form (2.3.2) for all functions f_a in a certain closed set $\mathfrak{C}_a$, it suffices to check the existence of the limit for the functions f_a in a set $\mathfrak{C}'_a$ which is everywhere dense in $\mathfrak{C}_a$. The result for the closed set $\mathfrak{C}_a$ then follows with the reasoning given in section 2.2.1. It also follows from that section that it is in fact sufficient to show that, for every f_a in $\mathfrak{C}'_a$ and every positive δ, there is a number T such that

$$||\Omega_a(s)f_a - \Omega_a(t)f_a|| < \delta \qquad (s,t < -T; s,t > T). \tag{2.4.1}$$

Here T may depend on f_a.

Let us now imagine that we want to establish eq. (2.4.1) for a function f_a in $\mathfrak{D}(H_0)$. If f_a belongs to $\mathfrak{D}(H_0)$, so does $\exp(-iH_a t)f_a$, since H_a commutes with H_0. The function $\exp(-iH_a t)f_a$ then also belongs to $\mathfrak{D}(H)$, by eq. (2.1.5). Owing to this, we have

$$\lim_{\tau\to 0} ||\frac{1}{\tau}[e^{iH(t+\tau)}e^{-iH_a(t+\tau)} - e^{iHt}e^{-iH_a t}]f_a - ie^{iHt}(H-H_a)e^{-iH_a t}f_a|| = 0 \tag{2.4.2}$$

(RIESZ and SZ.-NAGY (11) section 137). Hence, for every g in $\mathfrak{L}^2$,

$$\frac{d}{dt}(g, \Omega_a(t)f_a) = i(g, e^{iHt}[H-H_a]e^{-iH_a t}f_a), \tag{2.4.3}$$

$$(g, [\Omega_a(t) - \Omega_a(s)]f_a) = i\int_s^t (g, e^{iHu}[H-H_a]e^{-iH_a u}f_a)du. \tag{2.4.4}$$

The particular choice $g = [\Omega_a(t) - \Omega_a(s)]f_a$ yields

$$\left.\begin{aligned}||[\Omega_a(t)-\Omega_a(s)]f_a||^2 &= \int_s^t du\int_s^t dv(e^{iHv}[H-H_a]e^{-iH_av}f_a, e^{iHu}[H-H_a]e^{-iH_au}f_a)\\ &< \left[\int_s^t ||(H-H_a)e^{-iH_au}f_a||du\right]^2.\end{aligned}\right\}\qquad(2.4.5)$$

Hence, if f_a belongs to $\mathfrak{D}(H_0)$, a sufficient condition for eq. (2.4.1) to be satisfied is

$$\int_{-\infty}^{\infty}||(H-H_a)e^{-iH_at}f_a||dt < \infty. \qquad (2.4.6)$$

2.4.2. Sufficient conditions on the interaction

Using the notation of section 2.3.1, we proceed to derive sufficient conditions on the functions V_{ij} under which eq. (2.4.6) is satisfied for a suitable set of functions f_a. For the Hamiltonian H we write

$$H = \sum_{j=1}^{m'} H_{(j)}(\boldsymbol{x}_j) + H_a(\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1}) - \lambda_a + \sum_{p,q} V_{pq}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{2m-1}). \qquad (2.4.7)$$

We recall that m denotes the number of fragments into which the system is split when it is in channel a. The operators $H_{(j)}(\boldsymbol{x}_j)$ are the Hamiltonians of the m' fragments which consist of at least two particles each. The symbol $H_a(\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1})$ stands for the operator H_a defined in section 2.3.1. From the meaning of the various quantities involved it is obvious that the summation with respect to p and q must include only interactions between particles belonging to different fragments.

Let us now consider a particular term V_{pq}. In general, this depends on $\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1}$. On the other hand, as regards the coordinates $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{m'}$ it depends only on the internal coordinates of the fragments to which the particles p and q belong. Let us denote these by $\boldsymbol{x}_{j(p)}$ and $\boldsymbol{x}_{j(q)}$. Furthermore, let us recall that, if fragment $j(p)$ consists of $n_{j(p)}$ particles, the coordinate $\boldsymbol{x}_{j(p)}$ has $n_{j(p)}-1$ three-dimensional components. If these are denoted by $\boldsymbol{x}_{j(p),r}$, the corresponding components of $\boldsymbol{x}_{j(q)}$ by $\boldsymbol{x}_{j(q),s}$, the function V_{pq} in question is of the form

$$V_{pq}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{2m-1}) = V_{pq}\left(\sum_r d_r\boldsymbol{x}_{j(p),r} + \sum_s d_s\boldsymbol{x}_{j(q),s} + \sum_{j=m+1}^{2m-1} c_j\boldsymbol{x}_j\right). \qquad (2.4.8)$$

Here it will be understood that, if fragment $j(p)$ consists of only one particle, the term with $\boldsymbol{x}_{j(p)}$ may simply be dropped from eq. (2.4.8).

It is an essential point that at least one of the constants c_j does not vanish. If in particular $c_h \neq 0$, it is convenient to write symbolically

$$V_{pq}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{2m-1}) = V_{pq}(c\boldsymbol{x}_h + d\boldsymbol{x}_{j\neq h}) \qquad (m+1 \leq h \leq 2m-1). \qquad (2.4.9)$$

In the following the quantity $|\boldsymbol{x}_h|$ is denoted by x_h, and similarly for other vectors.

We now study the integral (2.4.6) for functions of the general product-form given by eq. (2.3.4). In line with the previous section, we restrict the function $f(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1})$ of eq. (2.3.4) to the set $\mathfrak{E}$ consisting of all linear combinations of functions

$$f_y(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}) = \prod_{j=m+1}^{2m-1} f_{y_j}(\boldsymbol{x}_j) \tag{2.4.10}$$

the Fourier transforms of which are of the form

$$\prod_{j=m+1}^{2m-1} \hat{f}_{y_j}(\boldsymbol{k}_j) = \prod_{j=m+1}^{2m-1} [k_{j1} k_{j2} k_{j3} \exp(-k_j^2 - i\boldsymbol{k}_j \cdot \boldsymbol{y}_j)]. \tag{2.4.11}$$

In this expression k_{j1}, k_{j2}, k_{j3} are the three components of the vector $\boldsymbol{k}_j$. The symbol k_j stands for $|\boldsymbol{k}_j|$. The vector $\boldsymbol{y}_j$ is a parameter the three components of which may take any finite values. With a view to one-channel scattering, the set of functions $\mathfrak{E}$ was introduced by KURODA (6). It follows from Wiener's theorem on the closure of the translations of a function in $\mathfrak{L}^2$ that $\mathfrak{E}$ is dense in the space $\mathfrak{L}^2(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1})$ (WIENER (21) section 15). More precisely, given a function $f(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1})$ in $\mathfrak{L}^2$ and a positive δ, there is an integer A, a set of constants a_α and a set of vectors $\boldsymbol{y}(\alpha)$ such that

$$\| f(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}) - \sum_{\alpha=1}^{A} a_\alpha f_{\boldsymbol{y}(\alpha)}(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}) \| < \delta. \tag{2.4.12}$$

It is obvious that all functions in $\mathfrak{E}$ belong to $\mathfrak{D}(H_0(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}))$. Furthermore, since $\varphi_{(j)}(\boldsymbol{x}_j)$ is an eigenfunction of $H_{(j)}(\boldsymbol{x}_j)$, it belongs to $\mathfrak{D}(H_{(j)}(\boldsymbol{x}_j))$. Hence, by the assumption that eq. (2.1.5) is satisfied, it belongs to $\mathfrak{D}(H_0(\boldsymbol{x}_j))$. From this it is easily seen that, if f_a is equal to $\prod_{j=1}^{m'} \varphi_{(j)}(\boldsymbol{x}_j)$ times a function in $\mathfrak{E}$, it belongs to $\mathfrak{D}(H_0(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{m'}, \boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}))$. In other words, all the functions f_a we consider in the present context belong to $\mathfrak{D}(H_0)$. They can therefore be used in the argument of the previous section.

Owing to our particular choice for the functions f_a, the study of eq. (2.4.6) reduces to a study of expressions of the form

$$\int_{-\infty}^{\infty} \| V_{pq}(c\boldsymbol{x}_h + d\boldsymbol{x}_{j \neq h}) \prod_{j=1}^{m'} \varphi_{(j)}(\boldsymbol{x}_j) \prod_{j=m+1}^{2m-1} [\exp[i\Delta(\boldsymbol{x}_j)t] f_{y_j}(\boldsymbol{x}_j)] \| dt. \tag{2.4.13}$$

If these are finite, it follows that there exist wave-operators $\Omega_{a\pm}$, the closed sets $\mathfrak{E}_a$ containing all functions of the form (2.3.4), where now $f(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1})$ may be any square-integrable function of its arguments.

With eq. (2.4.11) it is not difficult to evaluate $\exp[i\Delta(\boldsymbol{x}_j)t] f_{y_j}(\boldsymbol{x}_j)$ explicitly. This was done by KURODA (6), who showed that

$$|\exp[i\Delta(\boldsymbol{x}_j)t] f_{y_j}(\boldsymbol{x}_j)| < \text{const.} \frac{|\boldsymbol{x}_j - \boldsymbol{y}_j|^3}{(1+t^2)^{\frac{9}{4}}} \exp\left[-\frac{|\boldsymbol{x}_j - \boldsymbol{y}_j|^2}{4(1+t^2)}\right]. \tag{2.4.14}$$

From this it follows that, if ζ is in the interval $0 < \zeta < 1$, we may write

$$|\exp[i\Delta(\boldsymbol{x}_j)t]f_{\boldsymbol{y}_j}(\boldsymbol{x}_j)| < \text{const.}(1+t^2)^{-\frac{1}{2}-\frac{1}{4}\zeta}(1+|c\boldsymbol{x}_j - c\boldsymbol{y}_j|)^{-\frac{1}{2}+\frac{1}{2}\zeta}, \tag{2.4.15}$$

c being the constant used in eqs. (2.4.9) and (2.4.13).

Let us now assume that there is a ζ with $0 < \zeta < 1$ such that

$$\iiint |V_{pq}(c\boldsymbol{x}_h + d\boldsymbol{x}_{j\neq h})\varphi_{(j(p))}(\boldsymbol{x}_{j(p)})\varphi_{(j(q))}(\boldsymbol{x}_{j(q)})|^2(1+|c\boldsymbol{x}_h - c\boldsymbol{y}_h|)^{-1+\zeta}d\boldsymbol{x}_{j(p)}d\boldsymbol{x}_{j(q)}d\boldsymbol{x}_h < M(\boldsymbol{y}_h), \tag{2.4.16}$$

where $M(\boldsymbol{y}_h)$ is finite for every finite $\boldsymbol{y}_h$, and independent of the coordinates $\boldsymbol{x}_j$ on which the integral on the left may still depend. In evaluating the norm in eq. (2.4.13), it is then convenient to perform the integrations with respect to $\boldsymbol{x}_{j(p)}, \boldsymbol{x}_{j(q)}$, and $\boldsymbol{x}_h$ first. In an obvious notation, this yields

$$\left.\begin{aligned} &\|V_{pq}(c\boldsymbol{x}_h + d\boldsymbol{x}_{j\neq h})\prod_{j=1}^{m'}\varphi_{(j)}(\boldsymbol{x}_j)\prod_{j=m+1}^{2m-1}[\exp[i\Delta(\boldsymbol{x}_j)t]f_{\boldsymbol{y}_j}(\boldsymbol{x}_j)]\|_{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{m'},\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1}} \\ &\quad < \text{const.}[M(\boldsymbol{y}_h)]^{\frac{1}{2}}(1+t^2)^{-\frac{1}{2}-\frac{1}{4}\zeta}\prod_{\substack{j=1\\ j\neq j(p),\,j(q)}}^{m'}{}' \|\varphi_{(j)}(\boldsymbol{x}_j)\|_{\boldsymbol{x}_j}\prod_{\substack{j=m+1\\ j\neq h}}^{2m-1}{}' \|f_{\boldsymbol{y}_j}(\boldsymbol{x}_j)\|_{\boldsymbol{x}_j}. \end{aligned}\right\} \tag{2.4.17}$$

Hence in eq. (2.4.13) the integral with respect to t converges. From this it follows that there exist wave-operators $\Omega_{a\pm}$ whenever eqs. (2.4.16) and (2.1.5) are satisfied.

In obtaining this result, it was tacitly assumed that the fragments $j(p)$ and $j(q)$ consist of at least two particles each. If this is not so, we simply omit, say, $\varphi_{(j(p))}(\boldsymbol{x}_{j(p)})$ and the integration with respect to $\boldsymbol{x}_{j(p)}$ from eq. (2.4.16), and the argument can be carried through essentially unchanged.

In the case of scattering of only two fragments, $\boldsymbol{x}_h$ must necessarily be $\boldsymbol{x}_3$. There is no coordinate $\boldsymbol{x}_j$ on which the integral in eq. (2.4.16) might depend. Now for $\boldsymbol{y}_3 = 0$, the condition (2.4.16) implies

$$\iiint |V_{pq}(\sum_r d_r\boldsymbol{x}_{1,r} + \sum_s d_s\boldsymbol{x}_{2,s} + c_3\boldsymbol{x}_3)\varphi_{(1)}(\boldsymbol{x}_1)\varphi_{(2)}(\boldsymbol{x}_2)|^2(1+|c_3|x_3)^{-1+\zeta}d\boldsymbol{x}_1 d\boldsymbol{x}_2 d\boldsymbol{x}_3 < M. \tag{2.4.18}$$

If this holds true, it follows from the inequality

$$(1+|c_3\boldsymbol{x}_3 - c_3\boldsymbol{y}_3|)^{-1+\zeta} < (1+|c_3|x_3)^{-1+\zeta}(1+|c_3|y_3)^{1-\zeta} \tag{2.4.19}$$

that eq. (2.4.16) is also satisfied for general vectors $\boldsymbol{y}_3$, with

$$M(\boldsymbol{y}_3) = M(1+|c_3|\,y_3)^{1-\zeta}. \tag{2.4.20}$$

It is true that $M(\boldsymbol{y}_3)$ is not bounded uniformly in y_3, but this does not matter. For in using functions $f_{\boldsymbol{y}}(\boldsymbol{x})$ according to eq. (2.4.12), we can always restrict ourselves to a finite number of bounded vectors $\boldsymbol{y}_3$. Hence for two-fragment channels the simplified equation (2.4.18) already gives a sufficient condition for the existence of wave-operators.

In scattering problems in which three or more fragments are involved, V_{pq} depends in general not only on $\boldsymbol{x}_h$, but also on one or more of the remaining vectors $\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}$. Then the condition that there must be a bound $M(\boldsymbol{y}_h)$ independent of $\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}$ becomes significant. It implies in fact that $M(\boldsymbol{y}_h)$ must be independent of $\boldsymbol{y}_h$. Conversely, if $M(\boldsymbol{y}_h)$ is independent of $\boldsymbol{y}_h$, it follows that it cannot depend on any $\boldsymbol{x}_j$ either. It is therefore convenient to demand simply that eq. (2.4.16) be satisfied by a constant $M = M(\boldsymbol{y}_h)$.

For eq. (2.4.16) to hold true it is obviously sufficient if

$$\int [V_{pq}(\boldsymbol{X})]^2 (1 + |\boldsymbol{X} - \boldsymbol{Y}|)^{-1+\zeta} d^3\boldsymbol{X} < \text{const.} \tag{2.4.21}$$

for some ζ with $0 < \zeta < 1$, and every $\boldsymbol{Y}$. We recall that a relation of this form was already discussed at the end of section 2.1.2 in connection with the Hamiltonian being self-adjoint. It is clearly fulfilled if $V_{pq}(\boldsymbol{X})$ belongs to $\mathfrak{L}^2(\boldsymbol{X})$. More generally, for eq. (2.4.21) to hold true it is sufficient if there are positive constants R and η such that

$$\int_{X \leq R} [V_{pq}(\boldsymbol{X})]^2 d^3\boldsymbol{X} < \infty, \quad |V_{pq}(\boldsymbol{X})| < \text{const.}\, X^{-1-\eta} \quad (X > R). \tag{2.4.22}$$

If $\eta > \frac{1}{2}$, eq. (2.4.22) implies that $V_{pq}(\boldsymbol{X})$ belongs to $\mathfrak{L}^2(\boldsymbol{X})$. If $0 < \eta \leq \frac{1}{2}$, it is convenient to choose in eq. (2.4.21) $\zeta = \eta$. It is then a question of

$$\int_{X \leq R} [V_{pq}(\boldsymbol{X})]^2 (1 + |\boldsymbol{X} - \boldsymbol{Y}|)^{-1+\eta} d^3\boldsymbol{X} + \text{const.} \int_{X \geq R} X^{-2-2\eta} (1 + |\boldsymbol{X} - \boldsymbol{Y}|)^{-1+\eta} d^3\boldsymbol{X}, \tag{2.4.23}$$

which is bounded uniformly in $\boldsymbol{Y}$ in virtue of Hölder's inequality (Burkill (22) section 5.6). Hence, roughly speaking, it is sufficient if V_{pq} is locally square-integrable and falls off more rapidly than the Coulomb interaction. This is a generalization of the multi-channel result due to Hack (23), according to which it is sufficient if V_{pq} is square-integrable.

As for the scattering of two particles without internal coordinates, it is sufficient if

$$\int [V_{pq}(\boldsymbol{X})]^2 (1 + X)^{-1+\zeta} d^3\boldsymbol{X} < \infty \tag{2.4.24}$$

for some ζ with $0 < \zeta < 1$. This condition was found by Kuroda (6). It is a special case of eq. (2.4.18).

2.4.3. The set of asymptotic wave-functions

It follows from the previous section that, if the interaction satisfies suitable conditions, there exist wave-operators $\Omega_{a\pm}$. The set $\mathfrak{C}_a$ of asymptotic wave-functions f_a contains all functions of the form (2.3.4), where $f(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1})$ may be any square-integrable function of its arguments. Under the assumption that

$$\int [V_{pq}(\boldsymbol{X})]^2 (1 + |\boldsymbol{X} - \boldsymbol{Y}|)^{-3+\zeta} d^3\boldsymbol{X} < \text{const.} \tag{2.4.25}$$

for some positive ζ, uniformly in $\boldsymbol{Y}$, we now show that $\mathfrak{C}_a$ is not larger than the set of wave-functions of the form (2.3.4). The assumption (2.4.25) is very mild. Indeed, it is satisfied whenever $V_{pq}(\boldsymbol{X})$ is locally square-integrable and of the order $O(X^{-\eta})$ as X tends to ∞, with some positive η. It is much less stringent than the condition (2.4.21) for the existence of the wave-operators $\Omega_{a\pm}$. In the course of this section it will become clear that, if the operator $H_a'(\boldsymbol{x})$ is restricted to the general form (2.3.1), with some real λ_a, there are not more channels than we have considered thus far.

To prove our assertion, it is sufficient to show that, if f_a belongs to $\mathfrak{C}_a$ and to $\mathfrak{D}(H_0)$, $E_a(l)f_a$ is of the form (2.3.4) for every real l, $E_a(l)$ denoting the resolution of the identity associated with H_a. Now if f_a belongs to $\mathfrak{C}_a$, so does $H_aE_a(l)f_a$, by the end of section 2.2.3. Hence

$$\lim_{t\to\mp\infty} ||(H\Omega_{a\pm} - e^{iHt}e^{-iH_at}H_a)E_a(l)f_a|| = 0, \tag{2.4.26}$$

owing to eq. (2.2.30). Also, if R stands for the resolvent of H, the operator $R(\mu)H$ is bounded whenever μ is not real. Therefore

$$\lim_{t\to\mp\infty} ||R(\mu)H(\Omega_{a\pm} - e^{iHt}e^{-iH_at})E_a(l)f_a|| = 0. \tag{2.4.27}$$

Combining this with the previous equation yields

$$\lim_{t\to\mp\infty} ||R(\mu)e^{iHt}(H - H_a)e^{-iH_at}E_a(l)f_a|| = 0. \tag{2.4.28}$$

Hence, in view of eq. (2.4.7) and the fact that f_a belongs to $\mathfrak{D}(H_0)$,

$$\lim_{t\to\mp\infty} ||R(\mu)\left[\sum_{j=1}^{m'} H_{(j)}(\boldsymbol{x}_j) - \lambda_a + \sum_{p,q} V_{pq}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{2m-1})\right]e^{-iH_at}E_a(l)f_a|| = 0. \tag{2.4.29}$$

Using the symbolic notation of eq. (2.4.9), we now show that

$$\lim_{t\to\mp\infty} ||R(\mu)V_{pq}(c\boldsymbol{x}_h + d\boldsymbol{x}_{j\neq h})e^{-iH_at}E_a(l)f_a|| = 0. \tag{2.4.30}$$

Since $R(\mu)V_{pq}$ is a bounded operator, by eqs. (2.1.4) and (2.1.5), it is sufficient to prove that there is a set of functions f in $\mathfrak{D}(H_0)$ which is dense in $\mathfrak{L}^2$ and such that

$$\lim_{t\to\mp\infty} ||R(\mu)V_{pq}(c\boldsymbol{x}_h + d\boldsymbol{x}_{j\neq h})e^{-iH_at}f|| = 0 \tag{2.4.31}$$

for every f in the set. Let us therefore consider the set consisting of all linear combinations of functions of the form

$$f(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{m'},\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1}) = g(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{m'})\prod_{j=m+1}^{2m-1} f_{\boldsymbol{y}_j}(\boldsymbol{x}_j), \tag{2.4.32}$$

where g is any function in $\mathfrak{L}^2(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{m'})$ and $f_{\boldsymbol{y}_j}(\boldsymbol{x}_j)$ is the function introduced in eq. (2.4.10). This set has a subset in $\mathfrak{D}(H_0)$ which is dense in $\mathfrak{L}^2$. If $0 < \zeta < 1$,

$$|\exp[i\Delta(\boldsymbol{x}_h)t]f_{\boldsymbol{y}_h}(\boldsymbol{x}_h)| < \text{const.}(1+t^2)^{-\frac{1}{4}\zeta}(1+|c\boldsymbol{x}_h - c\boldsymbol{y}_h|)^{-\frac{3}{2}+\frac{1}{2}\zeta}, \qquad (2.4.33)$$

by eq. (2.4.14). Hence, in view of eq. (2.4.25),

$$\left.\begin{aligned} & ||V_{pq}(c\boldsymbol{x}_h + d\boldsymbol{x}_{j\neq h})e^{-iH_a t}f||^2_{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{m'},\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1}} \\ < \text{const.}(1+t^2)^{-\frac{1}{2}\zeta}||g||^2_{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{m'}} & \prod_{\substack{j=m+1\\ j\neq h}}^{2m-1}{}' ||f_{\boldsymbol{y}_j}(\boldsymbol{x}_j)||^2_{\boldsymbol{x}_j}\int[V_{pq}(c\boldsymbol{x}_h + d\boldsymbol{x}_{j\neq h})]^2(1+|c\boldsymbol{x}_h - c\boldsymbol{y}_h|)^{-3+\zeta}d\boldsymbol{x}_h \\ & < \text{const.}(1+t^2)^{-\frac{1}{2}\zeta}||g||^2_{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{m'}}. \end{aligned}\right\} \qquad (2.4.34)$$

From this it is obvious that eq. (2.4.31) holds true for every f in the set under discussion, and also for every f in $\mathfrak{D}(H_0)$.

Since $R_0(\mu)(H-\mu)$ is a bounded operator, again by eq. (2.1.5), it now follows from eq. (2.4.29) that

$$\lim_{t\to\mp\infty} ||R_0(\mu)\left[\sum_{j=1}^{m'} H_{(j)}(\boldsymbol{x}_j) - \lambda_a\right]e^{-iH_a t}E_a(l)f_a|| = 0. \qquad (2.4.35)$$

Hence, since H_a commutes both with $H_{(j)}(\boldsymbol{x}_j)$ and with $R_0(\mu)$,

$$||R_0(\mu)\left[\sum_{j=1}^{m'} H_{(j)}(\boldsymbol{x}_j) - \lambda_a\right]E_a(l)f_a|| = 0, \qquad (2.4.36)$$

from which it is obvious that

$$\left[\sum_{j=1}^{m'} H_{(j)}(\boldsymbol{x}_j) - \lambda_a\right]E_a(l)f_a = 0. \qquad (2.4.37)$$

This equation can only be fulfilled if λ_a is of the form given in eq. (2.3.3). Also, if the eigenvalues $\lambda_{(j)}$ are not degenerate, $E_a(l)f_a$ must be of the form (2.3.4). In the case of degenerate eigenvalues, $E_a(l)$ might be a linear combination of functions of the form (2.3.4). But if we then further specify the channel concept as was explained in eq. (2.3.5), the functions $E_a(l)f_a$ for the various channels a with the same λ_a are again restricted to the form (2.3.4). Since $E_a(l)f_a$ tends to f_a if l tends to ∞, the function f_a must also be of the form (2.3.4). Hence we may conclude that the set $\mathfrak{C}_a$ is not larger than the set of functions of the form (2.3.4). Since by the previous section it is not smaller, it follows that $\mathfrak{C}_a$ must be equal to the set (2.3.4). This result holds true whenever eqs. (2.1.5), (2.4.25), and the general sufficient condition (2.4.16) are fulfilled. If we restrict the operators $H_a(\boldsymbol{x})$ to the general form of eq. (2.3.1), there are no channels besides the ones we have considered in the foregoing.

2.4.4. The continuous spectrum

Now that we can use the theory of wave-operators, we can justify the statement of section 1.7.6 concerning the continuous spectrum of the Hamiltonian $H^{(n)}$ for a system consisting of n particles with square-integrable two-body interactions. In section 1.7.6 we considered a splitting of the system into fragments of n_1 and n_2 particles $(n_1+n_2 = n)$. The lower bounds of the spectra of $H^{(n_1)}$ and $H^{(n_2)}$ were denoted by $\Lambda^{(n_1)}$ and $\Lambda^{(n_2)}$, respectively. It was asserted that the continuous spectrum of $H^{(n)}$ runs from $M^{(n)}$ to ∞, with

$$M^{(n)} = \min(\Lambda^{(n_1)} + \Lambda^{(n_2)}, \Lambda^{(n-1)}) \qquad (n_1, n_2 \geq 2,\ n_1 + n_2 = n), \qquad (2.4.38)$$

the minimum being taken with respect to all possible splittings.

According to paper I, $M^{(n)}$ is not smaller than the minimum in eq. (2.4.38). To justify our assertion, we now show that $M^{(n)}$ is not larger either. The proof is based on the result of section 2.3.2, according to which the continuous spectrum of H contains the spectrum of H_a if, given a point l in the spectrum of H_a, there is a function f_a in $\mathfrak{C}_a$ such that l is not an interior point of an interval in which $E_a(l;f_a,f_a) =$ const. It is obvious that this condition is fulfilled for all channels considered in this paper. Indeed, H_a is nothing but $H_0(\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1}) + \lambda_a$, and f_a may contain any function in $\mathfrak{L}^2(\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1})$ as a factor.

For the purpose of the present section, it is convenient to call a splitting which yields the minimum in eq. (2.4.38) a minimum-splitting. It is also convenient to define $\Lambda^{(1)} = 0$, and to drop the restriction $n_1, n_2 \geq 2$. This formally makes $\Lambda^{(n-1)}$ equal to a quantity of the form $\Lambda^{(n_1)} + \Lambda^{(n_2)}$.

Let us now consider a minimum-splitting into n_1 and n_2 particles. The simplest situation arises if $\Lambda^{(n_1)}$ and $\Lambda^{(n_2)}$ are the eigenvalues of the bound states $\varphi_{(1)}(\boldsymbol{x}_1)$ and $\varphi_{(2)}(\boldsymbol{x}_2)$, respectively. Then it is useful to consider the scattering of two fragments in these bound states, the Hamiltonian H_a taking the form $H_0(\boldsymbol{x}_3) + \Lambda^{(n_1)} + \Lambda^{(n_2)}$. From the fact that the spectrum of this Hamiltonian H_a runs from $\Lambda^{(n_1)} + \Lambda^{(n_2)}$ to ∞, it follows that $M^{(n)}$ cannot exceed $\Lambda^{(n_1)} + \Lambda^{(n_2)}$. Combining this with the result that $M^{(n)}$ is not less than the minimum in eq. (2.4.38), we see that eq. (2.4.38) is fulfilled.

To cover the case that $\Lambda^{(n_1)}$ and $\Lambda^{(n_2)}$ do not both correspond to bound states, we proceed by induction. We first consider the scattering of two particles. For this $H_a = H_0(\boldsymbol{x}_1)$. Hence the continuous spectrum of $H^{(2)}$ coincides with the spectrum of H_0. Since this runs from 0 to ∞, it follows that $M^{(2)} = 0$. Formally we have $M^{(2)} = \Lambda^{(1)} + \Lambda^{(1)}$, hence eq. (2.4.38) holds true for $n = 2$.

As our next step we consider a minimum-splitting for three particles. This involves a lower bound $\Lambda^{(2)}$ which is equal either to 0 or to the eigenvalue of a bound state $\varphi_{(1)}(\boldsymbol{x}_1)$. In the latter case the choice $H_a = H_0(\boldsymbol{x}_2) + \Lambda^{(2)}$, corresponding to the scattering of the third particle by the bound fragment, tells us that eq. (2.4.38) is fulfilled. In case $\Lambda^{(2)} = 0$ we write $\Lambda^{(2)} = \Lambda^{(1)} + \Lambda^{(1)}$, and by considering the scattering of three unbound particles, $H_a = H_0(\boldsymbol{x}_1, \boldsymbol{x}_2)$, we obtain eq. (2.4.38) again.

Let us now assume that relations of the form (2.4.38) have been proved for $2,3,\ldots,n-1$ particles. To establish the desired result for n particles, we consider again a minimum-splitting into n_1 and n_2 particles. If $n_1 \geq 2$ and $\Lambda^{(n_1)}$ is not the eigenvalue of a bound state, $\Lambda^{(n_1)}$ must belong to the continuous spectrum of $H^{(n_1)}$. As a matter of fact, it must then be equal to $M^{(n_1)}$, since by definition it is the lower bound of the spectrum of $H^{(n_1)}$. Hence, by our assumption, there is a minimum-splitting

$$\Lambda^{(n_1)} = \Lambda^{(n_{11})} + \Lambda^{(n_{12})} \qquad (n_{11}+n_{12} = n_1). \tag{2.4.39}$$

If $n_{1i} \geq 2$ and the quantity $\Lambda^{(n_{1i})}$ is not the eigenvalue of a bound state, the fragment consisting of n_{1i} particles is split further. And so on. In this way we finally obtain a decomposition of the form

$$\Lambda^{(n_1)} + \Lambda^{(n_2)} = \sum_{j=1}^{m'} \Lambda^{(n'_j)} + (m-m')\Lambda^{(1)} \qquad \left(m-m' = n - \sum_{j=1}^{m'} n'_j\right), \tag{2.4.40}$$

each $\Lambda^{(n'_j)}$ being the eigenvalue of a bound state $\varphi_{(j)}(\boldsymbol{x}_j)$. If we now consider the scattering problem in which in the distant past the system was split into m' bound fragments in eigenstates $\varphi_{(j)}(\boldsymbol{x}_j)$, plus $m-m'$ single unbound particles, hence

$$H_a = H_0(\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1}) + \sum_{j=1}^{m'} \Lambda^{(n'_j)}, \tag{2.4.41}$$

we see that

$$M^{(n)} = \sum_{j=1}^{m'} \Lambda^{(n'_j)} = \Lambda^{(n_1)} + \Lambda^{(n_2)}, \tag{2.4.42}$$

as we wished to prove.

2.5. The wave-operators and the resolvent

2.5.1. The spectral resolution

Under the assumption that the interaction is such that there exist wave-operators $\Omega_{a\pm}$, we proceed to express the quantities $(g, \Omega_{a\pm} f_a)$ in terms of the resolvent of the Hamiltonian H, which is denoted by R. The resolvent corresponding to H_a is denoted by R_a, the resolution of the identity by E_a. In the following g may be any function in $\mathfrak{L}^2$, the function f_a is restricted to the form (2.3.4).

Since $E_a(K^2)f_a$ belongs to $\mathfrak{D}(H_0)$, it is convenient to consider the relation

$$f_a(\boldsymbol{x}) = \underset{K\to\infty}{\text{l.i.m.}}\, E_a(K^2) f_a(\boldsymbol{x}) = \prod_{j=1}^{m'} \varphi_{(j)}(\boldsymbol{x}_j) \underset{K\to\infty}{\text{l.i.m.}}\, E_a(K^2) f(\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1}). \tag{2.5.1}$$

Corresponding to this we have

$$(g, \Omega_{a\pm} f_a) = \lim_{K\to\infty} (g, \Omega_{a\pm} E_a(K^2) f_a). \tag{2.5.2}$$

Remembering eq. (2.2.37), we now write

$$(g, \Omega_{a\pm} f_a) = \mp \{\lim_{K\to\infty} \lim_{\varepsilon\to 0}\} \varepsilon \int_0^{\mp\infty} e^{-\varepsilon|t|}(g, e^{iHt} e^{-iH_a t} E_a(K^2) f_a) dt, \tag{2.5.3}$$

where the limit-symbol means that

$$\lim_{K\to\infty} \lim_{\varepsilon\to 0} = \lim_{\varepsilon\to 0} \lim_{K\to\infty}. \tag{2.5.4}$$

Since $E_a(K^2)f_a$ belongs to $\mathfrak{D}(H_0)$, it follows from arguments such as used in section 2.4.1 that in eq. (2.5.3) we may integrate by parts, with the result that

$$(g, [\Omega_{a\pm} - 1] f_a) = i\{\lim_{K\to\infty} \lim_{\varepsilon\to 0}\} \int_0^{\mp\infty} e^{-\varepsilon|t|}(g, e^{iHt}[H - H_a] e^{-iH_a t} E_a(K^2) f_a) dt. \tag{2.5.5}$$

For convenience we now define

$$V_a = H - \sum_{j=1}^{m'} H_{(j)}(\boldsymbol{x}_j) - H_a(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}) + \lambda_a = \sum_{p,q} V_{pq}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{2m-1}) \tag{2.5.6}$$

(cf. eq. (2.4.7)). Acting on functions $E_a(K^2)f_a$, the operator V_a has the same effect as the operator $H - H_a$.

For the following it is slightly inconvenient that the quantity $V_a \exp(-iHt)g$ need not belong to $\mathfrak{L}^2$. However, if λ is any non-real number, we may write

$$\left.\begin{aligned}(g, e^{iHt} V_a e^{-iH_a t} E_a(K^2) f_a) &= (g, e^{iHt} V_a [H - V_a - \lambda]^{-1} [H - V_a - \lambda] e^{-iH_a t} E_a(K^2) f_a) \\ &= (g, e^{iHt} V_a [H - V_a - \lambda]^{-1} e^{-iH_a t} [H_a - \lambda] E_a(K^2) f_a).\end{aligned}\right\} \tag{2.5.7}$$

Here $[H - V_a - \lambda]^{-1}$ stands for the resolvent of $H - V_a$. According to eq. (2.1.5), its range is $\mathfrak{D}(H_0)$. Hence $V_a[H - V_a - \lambda]^{-1}$ is a bounded operator, again by eq. (2.1.5). Denoting its adjoint by $\{[H - V_a - \bar{\lambda}]^{-1} V_a\}$, we obtain

$$(g, e^{iHt} V_a e^{-iH_a t} E_a(K^2) f_a) = (\{[H - V_a - \bar{\lambda}]^{-1} V_a\} e^{-iHt} g, e^{-iH_a t} [H_a - \lambda] E_a(K^2) f_a). \tag{2.5.8}$$

We can now apply the spectral-resolution formula (1.4.28) to $\exp(-iH_a t)$. This yields

$$\left.\begin{aligned}i \int_0^{\mp\infty} e^{-\varepsilon|t|}(g, e^{iHt} V_a e^{-iH_a t} E_a(K^2) f_a) dt &= \frac{1}{2\pi} \int_0^{\mp\infty} dt \lim_{U_\pm \to \pm\infty} \lim_{\zeta\to 0} \int_{U_-}^{U_+} A du, \\ A = e^{-\varepsilon|t| - iut}(\{[H - V_a - \bar{\lambda}]^{-1} V_a\} e^{-iHt} g, &[R_a(u + i\zeta) - R_a(u - i\zeta)][H_a - \lambda] E_a(K^2) f_a).\end{aligned}\right\} \tag{2.5.9}$$

It follows from eq. (1.4.9) that in the above expression the integral with respect to u is a bounded function of t, uniformly with respect to $U_\pm$ and ζ. In virtue of this, we have

$$\int_0^{\mp\infty} dt \lim_{U_\pm \to \pm\infty} \lim_{\zeta \to 0} \int_{U_-}^{U_+} du = \lim_{U_\pm \to \pm\infty} \lim_{\zeta \to 0} \int_0^{\mp\infty} dt \int_{U_-}^{U_+} du = \lim_{U_\pm \to \pm\infty} \lim_{\zeta \to 0} \int_{U_-}^{U_+} du \int_0^{\mp\infty} dt. \quad (2.5.10)$$

The shortest, if not the most subtle way of justifying this uses first the theorem of dominated convergence, next Fubini's theorem (BURKILL (22) sections 3.10, 5.4). This implies that the integrals are considered as Lebesgue integrals. A justification entirely within the framework of Riemann integration requires several continuity arguments. These can also be carried through without too much difficulty.

Now that we have eq. (2.5.9), we may skip the factors $[H - V_a - \lambda]^{-1}$ and $H_a - \lambda$. Next we can apply the spectral-resolution formula to $\exp(iHt)$. In virtue of a relation analogous to eq. (2.5.10), this yields the expression

$$\left.\begin{aligned} &\frac{1}{4\pi^2 i} \lim_{U_\pm \to \pm\infty} \lim_{\zeta \to 0} \int_{U_-}^{U_+} du \lim_{V_\pm \to \pm\infty} \lim_{\eta \to 0} \int_{V_-}^{V_+} dv \int_0^{\mp\infty} B dt, \\ &B = e^{-\varepsilon|t| - iut + ivt}(g, [R(v + i\eta) - R(v - i\eta)] V_a [R_a(u + i\zeta) - R_a(u - i\zeta)] E_a(K^2) f_a). \end{aligned}\right\} \quad (2.5.11)$$

The integration with respect to t can now be performed. Using the integral representation of the resolvent $R(v \pm i\eta)$, eq. (1.4.11), we finally obtain

$$\left.\begin{aligned} &(g, [\Omega_{a\pm} - 1] f_a) \\ &= -\frac{1}{2\pi i}\{\lim_{K\to\infty} \lim_{\varepsilon \to 0}\} \lim_{U_\pm \to \pm\infty} \lim_{\zeta \to 0} \int_{U_-}^{U_+} (g, R(u \pm i\varepsilon) V_a [R_a(u + i\zeta) - R_a(u - i\zeta)] E_a(K^2) f_a) du. \end{aligned}\right\} \quad (2.5.12)$$

It is shown in the next section that in this expression the integration with respect to u may in fact be restricted to the interval $\lambda_a \leq u \leq K^2$.

2.5.2. Auxiliary formulas

In the final formula of the previous section, it is often convenient to go over to the Fourier transform of the function $f(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1})$. To show why this is so, we first consider, in the notation of section 1.4.2,

$$\left.\begin{aligned} &(g, e^{iH_0 t} E_0(K^2) f) = \int_{-\infty}^{\infty} e^{iut} dE_0(u; g, E_0(K^2) f) = \int_{-\infty}^{\infty} e^{iut} d_u(g, E_0(u) E_0(K^2) f) \\ &= \int_{-\infty}^{K^2} e^{iut} d(g, E_0(u) f) = \int_{-\infty}^{K^2} e^{iut} dE_0(u; g, f) = \frac{1}{2\pi i} \lim_{\zeta \to 0} \int_0^{K^2} e^{iut}(g, [R_0(u + i\zeta) - R_0(u - i\zeta)] f) du \\ &= \frac{1}{\pi} \lim_{\zeta \to 0} \int_0^{K^2} e^{iut} du \int \bar{g}(\boldsymbol{x}) d\boldsymbol{x} \int [\operatorname{Im} G_0^{(m)}(\boldsymbol{x}, \boldsymbol{y}; u + i\zeta)] f(\boldsymbol{y}) d\boldsymbol{y}. \end{aligned}\right\} \quad (2.5.13)$$

According to eq. (1.2.17), the Green function has the form

$$G_0^{(m)}(\boldsymbol{x},\boldsymbol{y};\lambda) = \frac{i}{4}\left[\frac{\sqrt{\lambda}}{2\pi|\boldsymbol{x}-\boldsymbol{y}|}\right]^{\frac{3}{2}m-\frac{5}{2}} H^{(1)}_{\frac{3}{2}m-\frac{5}{2}}\left(\sqrt{\lambda}\,|\boldsymbol{x}-\boldsymbol{y}|\right). \tag{2.5.14}$$

It refers to $(3m-3)$-dimensional $\boldsymbol{x}$ and $\boldsymbol{y}$.

For the Fourier transform of $f(\boldsymbol{x})$ we use the notation

$$\hat{f}(\boldsymbol{k}) = (2\pi)^{-\frac{3}{2}m+\frac{3}{2}}\int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})d\boldsymbol{x}. \tag{2.5.15}$$

The quantity $|\boldsymbol{k}|$ is denoted by k, and similarly for other vectors.

It is explained below that

$$(2\pi)^{-\frac{3}{2}m+\frac{3}{2}}\int e^{i\boldsymbol{k}\cdot\boldsymbol{y}} G_0^{(m)}(\boldsymbol{x},\boldsymbol{y};\lambda)d\boldsymbol{y} = (2\pi)^{-\frac{3}{2}m+\frac{3}{2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{k^2-\lambda}. \tag{2.5.16}$$

Now $G_0^{(m)}(\boldsymbol{x},\boldsymbol{y};\lambda)$ depends only on $\boldsymbol{x}-\boldsymbol{y}$, and it belongs to $\mathfrak{L}(\boldsymbol{x}-\boldsymbol{y})$. Hence, by Parseval's formula and a theorem on resultants (Titchmarsh (24) theorem 65),

$$\int \bar{g}(\boldsymbol{x})d\boldsymbol{x}\int G_0^{(m)}(\boldsymbol{x},\boldsymbol{y};\lambda)f(\boldsymbol{y})d\boldsymbol{y} = \int \bar{\hat{g}}(\boldsymbol{k})\frac{1}{k^2-\lambda}\hat{f}(\boldsymbol{k})d\boldsymbol{k}. \tag{2.5.17}$$

Equation (2.5.13) can therefore be simplified to

$$(g, e^{iH_0t}E_0(K^2)f) = \frac{1}{\pi}\lim_{\zeta\to 0}\int_0^{K^2} e^{iut}du\int \bar{\hat{g}}(\boldsymbol{k})\frac{\zeta}{(k^2-u)^2+\zeta^2}\hat{f}(\boldsymbol{k})d\boldsymbol{k}. \tag{2.5.18}$$

Here $\int d\boldsymbol{k}$ is a Lebesgue integral. On the other hand, $\int du$ was originally meant to be a Riemann integral. It is convenient to consider it as a Lebesgue integral henceforth. This does not change its value. It makes it possible to use Fubini's theorem and the theorem of dominated convergence to justify that

$$\lim_{\zeta\to 0}\int_0^{K^2} du\int d\boldsymbol{k} = \int d\boldsymbol{k}\lim_{\zeta\to 0}\int_0^{K^2} du. \tag{2.5.19}$$

The limit with respect to ζ can now be performed. By the theory of Cauchy's singular integral (Titchmarsh (24) section 1.17), we have

$$\left.\begin{aligned}\frac{1}{\pi}\lim_{\zeta\to 0}\int_0^{K^2} e^{iut}\frac{\zeta}{(k^2-u)^2+\zeta^2}du &= e^{ik^2t} \qquad (0<k<K),\\ &= 0 \qquad (k>K).\end{aligned}\right\} \tag{2.5.20}$$

Hence we finally obtain

$$(g, e^{iH_0 t} E_0(K^2) f) = \int_{k<K} e^{ik^2 t} \bar{\hat{g}}(\boldsymbol{k}) \hat{f}(\boldsymbol{k}) d\boldsymbol{k}. \tag{2.5.21}$$

This shows that the Fourier transform of $E_0(K^2)f(\boldsymbol{x})$ is equal to $\hat{f}(\boldsymbol{k})$ if $0 < k < K$ and vanishes if $k > K$. Since $E_a(K^2)$ is nothing but $E_0(K^2 - \lambda_a)$ acting in the space $\mathfrak{L}^2(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1})$, the Fourier transform of $E_a(K^2) f(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1})$ is equal to $\hat{f}(\boldsymbol{k}_{m+1}, \ldots, \boldsymbol{k}_{2m-1})$ if $0 < k < (K^2 - \lambda_a)^{\frac{1}{2}}$ and vanishes if $k > (K^2 - \lambda_a)^{\frac{1}{2}}$. If the argument of eq. (2.5.13) is applied to eq. (2.5.9), it follows that in eqs. (2.5.9) to (2.5.12) the integration can be restricted to the interval $\lambda_a \leq u \leq K^2$.

It still remains to show that the Fourier transform of the Green function satisfies eq. (2.5.16). If $m = 2$, this is easily checked directly. Let us now assume that it has been proved for $m = m_1$ and $m = m_2$. We know from eq. (1.5.27) that

$$\left.\begin{aligned} Q &\equiv \int \exp[i\boldsymbol{k}_1 \cdot (\boldsymbol{y}_1 - \boldsymbol{x}_1) + i\boldsymbol{k}_2 \cdot (\boldsymbol{y}_2 - \boldsymbol{x}_2)] G_0^{(m_1+m_2-1)}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2; \lambda) d\boldsymbol{y}_1 d\boldsymbol{y}_2 \\ &= \frac{1}{2\pi i} \int \exp[i\boldsymbol{k}_1 \cdot (\boldsymbol{y}_1 - \boldsymbol{x}_1) + i\boldsymbol{k}_2 \cdot (\boldsymbol{y}_2 - \boldsymbol{x}_2)] d\boldsymbol{y}_1 d\boldsymbol{y}_2 \int_C G_0^{(m_2)}(\boldsymbol{x}_2, \boldsymbol{y}_2; \lambda - \sigma) G_0^{(m_1)}(\boldsymbol{x}_1, \boldsymbol{y}_1; \sigma) d\sigma, \end{aligned}\right\} \tag{2.5.22}$$

where C is a contour in the σ-plane such that the singularities of $G_0(\sigma)$ are on the right of it, and those of $G_0(\lambda - \sigma)$ on the left of it. Now $G_0(\boldsymbol{x}, \boldsymbol{y}; \lambda)$ is an integrable function of $\boldsymbol{y}$, by eq. (1.7.83). More generally, it follows from the estimate given in eq. (1.7.83) that, if C is chosen in a suitable way, the repeated integral in eq. (2.5.22) converges absolutely. It is again convenient to consider it as a Lebesgue integral. Then it follows immediately from Fubini's theorem that the order of integration may be inverted. Our assumption concerning the transforms for $m = m_1$ and $m = m_2$ thus yields

$$Q = \frac{1}{2\pi i} \int_C \frac{1}{(k_2^2 - \lambda + \sigma)(k_1^2 - \sigma)} d\sigma. \tag{2.5.23}$$

Integration with respect to σ now gives the desired transform for $m = m_1 + m_2 - 1$. Hence eq. (2.5.16) is satisfied generally, as we wished to show.

2.5.3. The wave-operators in momentum space

We now extend the use of Fourier transforms to eq. (2.5.12). In doing so, it is convenient to combine the coordinates $\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}$ into a $(3m-3)$-dimensional coordinate $\boldsymbol{x}_a$, the coordinates $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{m'}$ into a coordinate $\boldsymbol{x}_a'$, and to write

$$\prod_{j=1}^{m'} \varphi_{(j)}(\boldsymbol{x}_j) = \varphi_a(\boldsymbol{x}_a'). \tag{2.5.24}$$

To be explicit, we henceforth write m_a instead of m. We use the operator $P_a(X_a)$ defined by

$$\left.\begin{aligned} P_a(X_a)f(\boldsymbol{x}_a', \boldsymbol{x}_a) &= f(\boldsymbol{x}_a', \boldsymbol{x}_a) \qquad (x_a < X_a), \\ &= 0 \qquad (x_a > X_a). \end{aligned}\right\} \quad (2.5.25)$$

In terms of the operator $P_a(X_a)$, we have

$$\left.\begin{aligned} &\int_0^{\mp\infty} e^{-\varepsilon|t|}(g, e^{iHt}[H-H_a]e^{-iH_at}E_a(K^2)f_a)dt \\ &= \lim_{X_a\to\infty}\int_0^{\mp\infty} e^{-\varepsilon|t|}(g, e^{iHt}P_a(X_a)[H-H_a]e^{-iH_at}E_a(K^2)f_a)dt. \end{aligned}\right\} \quad (2.5.26)$$

If this relation is used in eq. (2.5.5), the reasoning of section 2.5.1 can be carried through essentially unchanged to show that in eq. (2.5.12) we may replace V_a by $V_aP_a(X_a)$, provided we take

$$\left\{\lim_{K\to\infty}\lim_{\varepsilon\to 0}\right\}\lim_{X_a\to\infty}\lim_{\zeta\to 0}\int_{\lambda_a}^{K^2} du. \quad (2.5.27)$$

We now recall that f_a is of the product-form (2.5.1). The operator $R_a(u \pm i\zeta)$ is nothing but $R_0(u-\lambda_a \pm i\zeta)$ acting only in $\mathfrak{L}^2(\boldsymbol{x}_a)$. The function $P_a(X_a)V_aR(u \mp i\varepsilon)g$ belongs to $\mathfrak{L}^2(\boldsymbol{x}_a', \boldsymbol{x}_a)$. If we denote it by $h(\boldsymbol{x}_a', \boldsymbol{x}_a)$, the quantity

$$\int \overline{\varphi}_a(\boldsymbol{x}_a')h(\boldsymbol{x}_a', \boldsymbol{x}_a)d\boldsymbol{x}_a' \quad (2.5.28)$$

is analogous to the function $\exp(-iut)g(\boldsymbol{x})$ in eq. (2.5.13). It is appropriate to denote the complex conjugate of its Fourier transform by

$$(2\pi)^{-\frac{3}{2}m_a+\frac{3}{2}}(g, R(u \pm i\varepsilon)V_aP_a(X_a)\varphi_a e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a}). \quad (2.5.29)$$

This is analogous to the function $\exp(iut)\bar{\hat{g}}(\boldsymbol{k})$ in eq. (2.5.18). Owing to the operator P_a, it does not exceed a constant depending on X_a, times the norm $||V_aR(u \mp i\varepsilon)g||$. Now it follows from eqs. (1.7.18) and (1.7.19) that, if each function V_{ij} satisfies eq. (2.1.5), the norm in question is bounded uniformly in u in the interval $\lambda_a \le u \le K^2$. Hence we may repeat the step (2.5.19). With the resolvent equation (1.2.11) it is now easily checked that the Fourier transform (2.5.29) is a continuous function of u. This makes it possible to perform the limit with respect to ζ as in eq. (2.5.20). The final result takes the form

$$\left.\begin{aligned} &(g, [\Omega_{a\pm}-1]f_a) \\ &= -(2\pi)^{-\frac{3}{2}m_a+\frac{3}{2}}\left\{\lim_{K\to\infty}\lim_{\varepsilon\to 0}\right\}\lim_{X_a\to\infty}\int_{k_a<K}(g, R(k_a^2+\lambda_a \pm i\varepsilon)V_aP_a(X_a)\varphi_a e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a})\hat{f}(\boldsymbol{k}_a)d\boldsymbol{k}_a. \end{aligned}\right\} \quad (2.5.30)$$

2.6. The scattering operators and the resolvent

2.6.1. General formulas

If both f_a and g_b are of the product-form (2.3.4), it is obvious that with the help of eq. (2.5.12) we can express $(g_b, \Omega^*_{b-}\Omega_{a+}f_a)$ in terms of the resolvent. A straightforward way of doing this yields a repeated integral, say $\int du \int dv$. A more interesting expression, with a single integral $\int du$, arises in the following way.

According to eq. (2.5.5) and the intertwining property (2.3.11), we have

$$\left.\begin{aligned} &(g_b, \Omega^*_{b-}[\Omega_{a+} - 1]f_a) \\ &= -i\{\lim_{K_a\to\infty}\lim_{\delta\to 0}\}\int_{-\infty}^{0} e^{\delta t}(g_b, e^{iH_bt}\Omega^*_{b-}V_a e^{-iH_at}E_a(K_a^2)f_a)dt \\ &= -i\{\lim_{K_a\to\infty}\lim_{\delta\to 0}\}\int_{-\infty}^{0} e^{\delta t}dt\Bigg[(g_b, e^{iH_bt}V_a e^{-iH_at}E_a(K_a^2)f_a) \\ &-i\{\lim_{K_b\to\infty}\lim_{\varepsilon\to 0}\}\int_{0}^{\infty} e^{-\varepsilon s}(E_b(K_b^2)g_b, e^{iH_b(s+t)}V_b e^{-iHs}V_a e^{-iH_at}E_a(K_a^2)f_a)ds\Bigg]. \end{aligned}\right\} \quad (2.6.1)$$

Now in eq. (2.5.3) the integral $\int dt$ is bounded uniformly in K and ε. Hence so is the integral $\int dt$ in eq. (2.5.5). Likewise, the integral $\int ds$ in the last term of eq. (2.6.1) is bounded uniformly in K_b and ε. Hence $\int dt$ and $\{\lim_{K_b} \lim_\varepsilon\}$ may be interchanged. With the techniques of sections 2.5.1 and 2.5.2 we thus obtain

$$(g_b, \Omega^*_{b-}[\Omega_{a+} - 1]f_a) = \{\lim_{K_a\to\infty}\lim_{\delta\to 0}\}[A + \{\lim_{K_b\to\infty}\lim_{\varepsilon\to 0}\}B], \quad (2.6.2)$$

where A and B are given by

$$\left.\begin{aligned} A &= -\frac{1}{2\pi i}\lim_{\zeta\to 0}\int_{\lambda_a}^{K_a^2}(g_b, R_b(u+i\delta)V_a[R_a(u+i\zeta) - R_a(u-i\zeta)]f_a)du \\ &= \frac{1}{2\pi i}\lim_{K_b\to\infty}\lim_{\zeta\to 0}\int_{\lambda_b}^{K_b^2}(g_b, [R_b(u+i\zeta) - R_b(u-i\zeta)]V_aR_a(u-i\delta)E_a(K_a^2)f_a)du, \\ B &= -\frac{1}{2\pi i}\lim_{\zeta\to 0}\int_{\lambda_b}^{K_b^2}(g_b, [R_b(u+i\zeta) - R_b(u-i\zeta)]V_bR(u+i\varepsilon)V_aR_a(u-i\delta)E_a(K_a^2)f_a)du. \end{aligned}\right\} \quad (2.6.3)$$

To obtain an expression which is more symmetric in a and b, we now consider $(g_b, \Omega^*_{b-}\Omega_{a-}f_a)$. According to eq. (2.3.23), this is equal to $\delta_{ba}(g_b, f_a)$. To express $(g_b, \Omega^*_{b-}\Omega_{a-}f_a)$ in terms of the resolvent, we merely have to replace $\int_{-\infty}^{0}\exp(\delta t)dt$ in eq. (2.6.1) by $-\int_0^\infty \exp(-\delta t)dt$. Corresponding to eq. (2.6.3), this yields relations with $-\delta$ instead of δ. Hence from the expression for $(g_b, \Omega_{b-}^*[\Omega_{a+}-\Omega_{a-}]f_a)$ it follows that

$$\left.\begin{aligned} (g_b, S_{ba}f_a) &= \delta_{ba}(g_b,f_a) + \{\lim_{K_a\to\infty}\lim_{\delta\to 0}\}[C + \{\lim_{K_b\to\infty}\lim_{\varepsilon\to 0}\}D],\\ C &= -\frac{1}{2\pi i}\lim_{K_b\to\infty}\lim_{\zeta\to 0}\int_{\lambda_b}^{K_b^2}(g_b,[R_b(u+i\zeta)-R_b(u-i\zeta)]V_a[R_a(u+i\delta)-R_a(u-i\delta)]E_a(K_a^2)f_a)du,\\ D &= \frac{1}{2\pi i}\lim_{\zeta\to 0}\int_{\lambda_b}^{K_b^2}(g_b,[R_b(u+i\zeta)-R_b(u-i\zeta)]V_bR(u+i\varepsilon)V_a[R_a(u+i\delta)-R_a(u-i\delta)]E_a(K_a^2)f_a)du. \end{aligned}\right\} \quad (2.6.4)$$

As regards the quantity C, we observe that, owing to eqs. (2.3.17) and (2.3.20),

$$\mp\lim_{\delta\to 0}\delta\int_0^{\mp\infty}e^{-\delta|t|}(g_b, e^{iH_bt}e^{-iH_at}f_a)dt = \delta_{ba}(g_b,f_a). \quad (2.6.5)$$

Upon integrating by parts, this yields

$$(\delta_{ba}-1)(g_b,f_a) = i\{\lim_{K_a\to\infty}\lim_{\delta\to 0}\}\int_0^{\mp\infty}e^{-\delta|t|}(g_b, e^{iH_bt}[H_b-H_a]e^{-iH_at}E_a(K_a^2)f_a)dt. \quad (2.6.6)$$

In this expression the operator $H_b - H_a$ may be replaced by $V_a - V_b$, by eq. (2.5.6). Hence

$$\left.\begin{aligned} &(\delta_{ba}-1)(g_b,f_a)\\ &= \frac{1}{2\pi i}\{\lim_{K_a\to\infty}\lim_{\delta\to 0}\}\lim_{K_b\to\infty}\lim_{\zeta\to 0}\int_{\lambda_b}^{K_b^2}(g_b,[R_b(u+i\zeta)-R_b(u-i\zeta)][V_a-V_b]R_a(u\mp i\delta)E_a(K_a^2)f_a)du, \end{aligned}\right\} \quad (2.6.7)$$

from which it follows that

$$\left.\begin{aligned} &\{\lim_{K_a\to\infty}\lim_{\delta\to 0}\}\lim_{K_b\to\infty}\lim_{\zeta\to 0}\int_{\lambda_b}^{K_b^2}(g_b,[R_b(u+i\zeta)-R_b(u-i\zeta)]\\ &\times[V_a-V_b][R_a(u+i\delta)-R_a(u-i\delta)]E_a(K_a^2)f_a)du = 0. \end{aligned}\right\} \quad (2.6.8)$$

Combining this with eq. (2.6.4), we see that in the expression for C the operator V_a may be replaced by $(V_a+V_b)/2$. This increases the symmetry between a and b. However, as for the quantity D, there is still a fundamental asymmetry owing to the fact that $\lim_\varepsilon$ must be performed before $\lim_\delta$. This prevents us from going over to the

Fourier transform of f_a. It seems that this cannot be changed unless we make additional assumptions on the function f_a and on the interaction V_a. This point is the subject of the following sections.

2.6.2. The repeated limit

Equation (2.6.4) essentially results from evaluating

$$\left.\begin{aligned} &\lim_{\delta\to 0}\int_{-\infty}^{\infty} e^{-\delta|t|}(g_b, e^{iH_bt}\Omega_{b-}^{*}V_a e^{-iH_at}E_a(K_a^2)f_a)dt \\ &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} e^{-\delta|t|}(g_b, e^{iH_bt}\Omega_{b-,\varepsilon}^{*}V_a e^{-iH_at}E_a(K_a^2)f_a)dt, \end{aligned}\right\} \quad (2.6.9)$$

where $\Omega_{b-,\varepsilon}$ is an obvious notation for an operator analogous to the quantity (2.2.39). Now we should like to interchange the two limits in eq. (2.6.9). Going over to the Fourier transform of f_a, we could then perform the limit with respect to δ by the method of section 2.5.3. Interchanging the limits is permitted if and only if

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{-\infty}^{\infty} e^{-\delta|t|}([\Omega_{b-,\varepsilon}-\Omega_{b-}]e^{-iH_bt}g_b, V_a e^{-iH_at}E_a(K_a^2)f_a)dt = 0. \quad (2.6.10)$$

Since $E_a(K_a^2)f_a$ belongs to $\mathfrak{D}(H_0)$, it is not difficult to see that the inner product in eq. (2.6.10) is a bounded and continuous function of t. Hence, if $\delta > 0$, the integral certainly exists. But it is not at all obvious that it tends to a limit as δ tends to 0. However, let us for a moment drop the operator $E_a(K_a^2)$, let us assume that V_a satisfies the sufficient condition (2.4.16), and let us consider the special set of functions f_a introduced in section 2.4.2. Denoting this set by $\mathfrak{E}_a$, we see from eq. (2.4.17) that, if f_a is any particular function in $\mathfrak{E}_a$, there exist numbers $N(f_a)$ and ζ, with $0 < \zeta < 1$, such that

$$\|V_a e^{-iH_at}f_a\| < N(f_a)(1+t^2)^{-\frac{1}{2}-\frac{1}{4}\zeta}. \quad (2.6.11)$$

Hence the integrand in eq. (2.6.10) does not exceed an integrable function of t. As a result $\lim_\delta$ and $\int dt$ may be interchanged, by the theorem of dominated convergence. Also,

$$\left.\begin{aligned} &\int_{-\infty}^{\infty}\|(\Omega_{b-,\varepsilon}-\Omega_{b-})e^{-iH_bt}g_b\|\,\|V_a e^{-iH_at}f_a\|dt \\ &< N(f_a)\int_{-T}^{T}\|(\Omega_{b-,\varepsilon}-\Omega_{b-})e^{-iH_bt}g_b\|dt + 2N(f_a)\|g_b\|\left(\int_{-\infty}^{-T}+\int_{T}^{\infty}\right)(1+t^2)^{-\frac{1}{2}-\frac{1}{4}\zeta}dt. \end{aligned}\right\} \quad (2.6.12)$$

As T tends to ∞, the second term on the right clearly tends to 0. Given T, it follows from an analogue of eq. (2.2.39) that

$$
\left.\begin{aligned}
I &\equiv \int_{-T}^{T} ||(\Omega_{b-,\varepsilon} - \Omega_{b-})e^{-iH_b t} g_b|| dt \\
&\leq \int_{-T}^{T} ||e^{\varepsilon t} e^{-iHt} \Omega_{b-,\varepsilon} g_b + \varepsilon e^{\varepsilon t} e^{-iHt} \int_{t}^{0} e^{-\varepsilon s} \Omega_b(s) ds g_b - e^{-iHt} \Omega_{b-} g_b|| dt \\
&\leq \int_{-T}^{T} [||(\Omega_{b-,\varepsilon} - \Omega_{b-}) g_b|| + ||g_b|| \, |e^{\varepsilon t} - 1| + \varepsilon ||g_b|| e^{\varepsilon t} |\int_{t}^{0} e^{-\varepsilon s} ds|] dt \\
&\leq 2T ||(\Omega_{b-,\varepsilon} - \Omega_{b-}) g_b|| + 4(\cosh \varepsilon T - 1) ||g_b|| / \varepsilon.
\end{aligned}\right\} \quad (2.6.13)
$$

Hence, if given T and a positive η, we choose ε so small that

$$
2T||(\Omega_{b-,\varepsilon} - \Omega_{b-})g_b|| < \eta, \qquad 4(\cosh \varepsilon T - 1)||g_b||/\varepsilon < \eta, \qquad (2.6.14)
$$

the quantity I does not exceed 2η. From this it follows that by choosing first T sufficiently large, next ε sufficiently small, the right-hand side of eq. (2.6.12) can be made arbitrarily close to 0. Hence eq. (2.6.10) is satisfied if we drop the operator $E_a(K_a^2)$ and for f_a take a function in $\mathfrak{E}_a$. For future reference we note that this result is simply due to $||V_a \exp(-iH_a t) f_a||$ being a bounded and integrable function of t.

Unfortunately, it does not follow from the foregoing that eq. (2.6.10) holds true generally. A conclusion to that effect could be drawn if it were known, for instance, that the limit with respect to δ existed and were bounded by a constant times $||f_a||$, uniformly in ε. But a result of this sort is not available. There is another difficulty, which is related to the operator $E_a(K_a^2)$. Throughout this investigation, we find it convenient to have functions of the form $E_a(K_a^2)f_a$, i. e. functions whose Fourier transforms with respect to $\boldsymbol{x}_a$ vanish outside bounded regions. Now the functions in $\mathfrak{E}_a$ are not of the desired form. Hence they cannot really be used in the present context. We have not succeeded in deciding whether there is a suitable set of functions $E_a(K_a^2)f_a$ such that eq. (2.6.10) holds true whenever V_a satisfies the relation (2.4.16). What we do know is that eq. (2.6.10) is valid for fairly large classes of interactions V_a and functions $E_a(K_a^2)f_a$. We now pass on to discussing this.

2.6.3. Restrictions on the interaction and on the relative motion

In the present section we formulate sufficient conditions on the interaction V_a and on the wave-function f_a to guarantee that in eqs. (2.6.1) and (2.6.4) the limits with respect to ε and δ may be interchanged. The functions f_a we consider all have Fourier transforms with respect to $\boldsymbol{x}_a$ which vanish outside bounded regions. Hence, if K_a is large enough, $E_a(K_a^2)f_a$ is nothing but f_a, and we need not distinguish between the two quantities.

If the limits are interchangeable for any particular combination V_a, f_a, we say that this combination is admissible. It is clear from eq. (2.6.1) that, if both V_{a1}, f_a and V_{a2}, f_a are admissible, so is $V_{a1} + V_{a2}, f_a$. Hence we may restrict the discussion to certain standard forms of the interaction V_a, it being understood that a linear combination of these can be considered without additional difficulties.

As in previous sections, the function f_a will be of the form

$$f_a(\boldsymbol{x}) = \prod_{j=1}^{m'} \varphi_{(j)}(\boldsymbol{x}_j) f(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}). \tag{2.6.15}$$

Now let $\mathfrak{F}$ denote the set of all functions $f(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1})$ in $\mathfrak{L}^2$ the Fourier transforms of which have bounded first-order partial derivatives with respect to $\boldsymbol{k}_{m+1}, \ldots, \boldsymbol{k}_{2m-1}$ and vanish if $k^2_{m+1} + \ldots + k^2_{2m-1} > K^2$, the parameter K running through all finite values. Then we assume in the following that f belongs to $\mathfrak{F}$. If f belongs to $\mathfrak{F}$ and f_a is of the form (2.6.15), we say that f_a belongs to $\mathfrak{F}_a$.

It is not difficult to see that $\mathfrak{F}$ is dense in $\mathfrak{L}^2(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1})$. Indeed, the Fourier transform of any function in $\mathfrak{L}^2(\boldsymbol{x})$ belongs to $\mathfrak{L}^2(\boldsymbol{k})$. It can be approximated in mean square by a continuous function which vanishes outside a bounded region (McShane (25) section 42.4s). Such a continuous function can be approximated uniformly by a function which vanishes outside a bounded region and has continuous partial derivatives of all orders (Schwartz (26) ch. I, theorem I). Hence the set of Fourier transforms of functions in $\mathfrak{F}$ is dense in $\mathfrak{L}^2(\boldsymbol{k}_{m+1}, \ldots, \boldsymbol{k}_{2m-1})$. As a result $\mathfrak{F}$ is dense in $\mathfrak{L}^2(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1})$. The set $\mathfrak{F}_a$ is dense in the set of functions (2.6.15).

We now turn to the function V_a. This is a sum of two-body interactions V_{pq}, by eq. (2.5.6). We saw in eq. (2.4.8) that each V_{pq} depends on the internal coordinates $\boldsymbol{x}_{j(p)}$ of the fragment to which particle p belongs, similarly on the coordinates $\boldsymbol{x}_{j(q)}$ ($j(p) \neq j(q); j(p), j(q) = 1, 2, \ldots, m'$), and on one or more coordinates $\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}$. For the major part of the following sections, we concentrate on one particular V_{pq}. Given p and q, we choose the coordinates $\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}$ in such a way that there is a certain $\boldsymbol{x}_h$ ($h = m+1, \ldots, 2m-1$) proportional to the distance between the centres of mass of the fragments $j(p)$ and $j(q)$. This yields

$$V_{pq}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{2m-1}) = V_{pq}(\sum_r d_r \boldsymbol{x}_{j(p),r} + \sum_s d_s \boldsymbol{x}_{j(q),s} + c\boldsymbol{x}_h) \tag{2.6.16}$$

(cf. eq. (2.4.8)).

The restrictions we want to impose upon V_{pq} are most easily expressed in terms of the function

$$W_{pq}(\boldsymbol{x}_h) = \iint |V_{pq}(\sum_r d_r \boldsymbol{x}_{j(p),r} + \sum_s d_s \boldsymbol{x}_{j(q),s} + c\boldsymbol{x}_h) \varphi_{(j(p))}(\boldsymbol{x}_{j(p)}) \varphi_{(j(q))}(\boldsymbol{x}_{j(q)})|^2 d\boldsymbol{x}_{j(p)} d\boldsymbol{x}_{j(q)}, \tag{2.6.17}$$

appropriate modifications being understood in case the fragments $j(p)$ and $j(q)$ consist of only one particle. We assume that W_{pq} is integrable, hence that V_{pq} is a square-integrable function of $\boldsymbol{x}_h$. Then the general sufficient condition of section 2.4.2 is certainly fulfilled. For the time being we further restrict ourselves to the case that

$$\int [W_{pq}(\boldsymbol{x}_h) x_h^2]^{\mu} d\boldsymbol{x}_h < \infty \tag{2.6.18}$$

for some μ with $1 < \mu < 2$. Slightly more general classes of functions W_{pq} are considered in section 2.6.5.

2.6.4. A convergence problem

It follows from section 2.6.2 that for V_a, f_a to be admissible, it is sufficient if $||V_a \exp(-iH_a t) f_a||$ is a bounded and integrable function of t. Assuming that f_a belongs to $\mathfrak{F}_a$, we now show that the particular function V_{pq} we are considering satisfies

$$\int_{-\infty}^{\infty} ||V_{pq} e^{-iH_a t} f_a|| dt < \infty. \tag{2.6.19}$$

Since $\mathfrak{F}_a$ is contained in $\mathfrak{D}(H_0)$, the integrand in eq. (2.6.19) is a bounded and continuous function of t. Hence the integration over the interval $-1 < t < 1$ does not present difficulties. To cope with the case $|t| > 1$, we write

$$\left.\begin{aligned} &\exp\left[i \sum_{j=m+1}^{2m-1} \Delta(\boldsymbol{x}_j) t\right] f(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}) \\ &= (2\pi)^{-\frac{3}{2}m+\frac{3}{2}} \int \exp\left[i \sum_{j=m+1}^{2m-1} (-k_j^2 t + \boldsymbol{k}_j \cdot \boldsymbol{x}_j)\right] \hat{f}(\boldsymbol{k}_{m+1}, \ldots, \boldsymbol{k}_{2m-1}) d\boldsymbol{k}_{m+1} \ldots d\boldsymbol{k}_{2m-1}. \end{aligned}\right\} \tag{2.6.20}$$

Integration by parts with respect to k_h yields

$$\left.\begin{aligned} &\exp\left[i \sum_{j=m+1}^{2m-1} \Delta(\boldsymbol{x}_j) t\right] f(\boldsymbol{x}_{m+1}, \ldots, \boldsymbol{x}_{2m-1}) \\ &= \frac{(2\pi)^{-\frac{3}{2}m+\frac{3}{2}}}{2it} \int \exp\left[i \sum_{j=m+1}^{2m-1} (-k_j^2 t + \boldsymbol{k}_j \cdot \boldsymbol{x}_j)\right] \\ &\times (i\boldsymbol{k}_h \cdot \boldsymbol{x}_h + 1 + \boldsymbol{k}_h \cdot \mathrm{grad}_h) \hat{f}(\boldsymbol{k}_{m+1}, \ldots, \boldsymbol{k}_{2m-1}) k_h^{-2} d\boldsymbol{k}_{m+1} \ldots d\boldsymbol{k}_{2m-1}, \end{aligned}\right\} \tag{2.6.21}$$

the integrated term vanishing since $\hat{f}$ vanishes if $k_h > K$.

We first consider the term proportional to $i\boldsymbol{k}_h \cdot \boldsymbol{x}_h$. Owing to our special choice of coordinates, we have

$$\left.\begin{aligned} I_1(t) &= ||V_{pq}(\boldsymbol{x}) \prod_{j=1}^{m'} \varphi_{(j)}(\boldsymbol{x}_j) \int \exp\left[i \sum_{j=m+1}^{2m-1} (-k_j^2 t + \boldsymbol{k}_j \cdot \boldsymbol{x}_j)\right] \\ &\times \boldsymbol{k}_h \cdot \boldsymbol{x}_h \hat{f}(\boldsymbol{k}_{m+1}, \ldots, \boldsymbol{k}_{2m-1}) k_h^{-2} d\boldsymbol{k}_{m+1} \ldots d\boldsymbol{k}_{2m-1}||^2 \\ &= \int W_{pq}(\boldsymbol{x}_h) | \int \exp[i(-k_h^2 t + \boldsymbol{k}_h \cdot \boldsymbol{x}_h)] \boldsymbol{k}_h \cdot \boldsymbol{x}_h F(\boldsymbol{k}_h) k_h^{-2} d\boldsymbol{k}_h|^2 d\boldsymbol{x}_h, \end{aligned}\right\} \tag{2.6.22}$$

where F is a bounded function of $\boldsymbol{k}_h$ which vanishes if $k_h > K$. It is an essential point that F does not depend on t.

In the rest of the analysis, we drop the subscript h. On the other hand, we consider separately the three components of $\boldsymbol{k}_h$ and $\boldsymbol{x}_h$, which we denote by k_γ and x_γ $(\gamma = 1,2,3)$.

If in eq. (2.6.18) $\mu = 1$, the function

$$Z_{\beta\gamma}(\boldsymbol{k}-\boldsymbol{k}') = (2\pi)^{-\frac{3}{2}}\int W_{pq}(\boldsymbol{x})\exp[i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}]x_\beta x_\gamma d\boldsymbol{x} \qquad (2.6.23)$$

is bounded. If $1 < \mu \leq 2$, it satisfies

$$\left.\begin{aligned}&\left[(2\pi)^{-\frac{3}{2}}\int |Z_{\beta\gamma}(\boldsymbol{k}-\boldsymbol{k}')|^{\mu/(\mu-1)}d(\boldsymbol{k}-\boldsymbol{k}')\right]^{(\mu-1)/\mu}\\ &\qquad < \left[(2\pi)^{-\frac{3}{2}}\int |W_{pq}(\boldsymbol{x})x_\beta x_\gamma|^{\mu}d\boldsymbol{x}\right]^{1/\mu}.\end{aligned}\right\} \qquad (2.6.24)$$

The corresponding relation for Fourier transforms depending on one-dimensional variables k and x was proved by Titchmarsh (24) (sections 4.1–4.5) and, by a different method, by Zygmund (27) (ch. XVI, section 3). Zygmund's proof, which is based on the Riesz-Thorin convexity theorem (Zygmund (27) ch. XII, section 1), applies to functions of several variables essentially unchanged.

With the help of eq. (2.6.24) it is easily shown that in eq. (2.6.22) the integration with respect to $\boldsymbol{x}_h$ may be performed first. Hence

$$I_1(t) = (2\pi)^{\frac{3}{2}}\sum_{\beta,\gamma}\iint Z_{\beta\gamma}(\boldsymbol{k}-\boldsymbol{k}')\exp[-i(k^2-k'^2)t]k_\beta k'_\gamma F(\boldsymbol{k})\bar{F}(\boldsymbol{k}')k^{-2}k'^{-2}d\boldsymbol{k}d\boldsymbol{k}'. \qquad (2.6.25)$$

We now go over to new variables as follows. First we write

$$\boldsymbol{k}-\boldsymbol{k}' = \boldsymbol{w}, \qquad \boldsymbol{k}+\boldsymbol{k}' = \boldsymbol{v}. \qquad (2.6.26)$$

Next we consider $\boldsymbol{v}$ in a coordinate frame fixed to $\boldsymbol{w}$. In this frame we introduce polar coordinates v,ϑ,φ, choosing the direction of $\boldsymbol{w}$ as the polar axis. Finally we go over from $\boldsymbol{w},v,\vartheta,\varphi$ to $\boldsymbol{w},v,\varphi$, $r = vw\cos\vartheta$, where w stands for $|\boldsymbol{w}|$. With this choice of coordinates we have

$$\left.\begin{aligned}k^2-k'^2 = \boldsymbol{v}.\boldsymbol{w} = r, \qquad 2k &= |\boldsymbol{v}+\boldsymbol{w}| = (v^2+w^2+2r)^{\frac{1}{2}},\\ 2k' &= |\boldsymbol{v}-\boldsymbol{w}| = (v^2+w^2-2r)^{\frac{1}{2}}.\end{aligned}\right\} \qquad (2.6.27)$$

The quantity I_1 therefore takes the form

$$I_1(t) = \sum_{\beta,\gamma}\int_{-K^2}^{K^2}dr\int_{w=\frac{|r|}{2K}}^{w=2K}d\boldsymbol{w}\int_{\frac{|r|}{w}}^{2K}dv\int_0^{2\pi}d\varphi\, e^{-irt}Z_{\beta\gamma}(\boldsymbol{w})w^{-1}v[(v^2+w^2)^2-4r^2]^{-\frac{1}{2}}G_{\beta\gamma}(\boldsymbol{w},v,\varphi,r), \qquad (2.6.28)$$

$G_{\beta\gamma}$ being some bounded function. Now if $w < 2K$,

$$\left.\begin{aligned}&\int_{\frac{|r|}{w}}^{2K} v[(v^2+w^2)^2-4r^2]^{-\frac{1}{2}}dv = \tfrac{1}{2}\log(4K^2+w^2+[(4K^2+w^2)^2-4r^2]^{\frac{1}{2}})\\ &-\tfrac{1}{2}\log\left(\frac{r^2}{w^2}+w^2+\left|\frac{r^2}{w^2}-w^2\right|\right) < \log 4K - \tfrac{1}{4}\log\frac{2r^2}{w^2} - \tfrac{1}{4}\log 2w^2 = \tfrac{1}{2}\log\frac{8K^2}{|r|}.\end{aligned}\right\} \qquad (2.6.29)$$

Hence

$$I_1(t) = \sum_{\beta,\gamma} \int_{-K^2}^{K^2} dr \int_{w=\frac{|r|}{2K}}^{w=2K} d\boldsymbol{w}\, e^{-irt} \log\frac{8K^2}{|r|} Z_{\beta\gamma}(\boldsymbol{w}) w^{-1} H_{\beta\gamma}(\boldsymbol{w}, r), \tag{2.6.30}$$

with some bounded function $H_{\beta\gamma}$.

It follows from eq. (2.6.24) that, if $1 < \mu < 2$,

$$\int_{w=\frac{|r|}{2K}}^{w=2K} |Z_{\beta\gamma}(\boldsymbol{w})|\, w^{-1} d\boldsymbol{w} < \left[\int |Z_{\beta\gamma}(\boldsymbol{w})|^{\mu/(\mu-1)} d\boldsymbol{w}\right]^{(\mu-1)/\mu} \left[\int_{w=\frac{|r|}{2K}}^{w=2K} w^{-\mu} d\boldsymbol{w}\right]^{1/\mu} < \infty. \tag{2.6.31}$$

If $\mu = 1$, the left-hand side of eq. (2.6.31) is likewise finite. Hence $I_1(t)$ is bounded. Also, $I_1(t)$ is the Fourier transform of a function of r which belongs to $\mathfrak{L}^{\nu}(r)$ $(1 < \nu \leq 2)$. As a result $I_1(t)$ belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)$. Hence $[I_1(t)]^{\frac{1}{2}}/t$ is integrable over $-\infty < t < -1$ and $1 \leq t < \infty$, by Hölder's inequality. In view of eq. (2.6.22), this shows that in eq. (2.6.21) the term proportional to $i\boldsymbol{k}_h . \boldsymbol{x}_h$ is compatible with eq. (2.6.19).

The term proportional to $\boldsymbol{k}_h .\mathrm{grad}_h$ can be discussed along the same lines. Instead of eq. (2.6.18) we merely need the fact that W_{pq} is integrable. For the term proportional to 1 we have to consider

$$\left.\begin{aligned} \int_{\frac{|r|}{2K}}^{2K} dw \int_{\frac{|r|}{w}}^{2K} wv[(v^2+w^2)^2 - 4r^2]^{-1} dv &= \frac{1}{8r}\int_{\frac{|r|}{2K}}^{2K} w \log\frac{(4K^2+w^2-2r)(w^2+r)^2}{(4K^2+w^2+2r)(w^2-r)^2} dw \\ &= \frac{1}{8r}[(4K^2+r)\log|4K^2+r| - (4K^2-r)\log|4K^2-r| - 2r\log 2|r|]. \end{aligned}\right\} \tag{2.6.32}$$

Now since

$$\lim_{r\to 0}\frac{1}{r}(\log|4K^2+r| - \log|4K^2-r|) = \frac{1}{2K^2}, \tag{2.6.33}$$

the right-hand side of eq. (2.6.32) has only logarithmic singularities. Hence, by analogy with the function $I_1(t)$, the term proportional to 1 yields a bounded function $I_2(t)$ which again belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)$ and has the property that $[I_2(t)]^{\frac{1}{2}}/t$ is integrable over $-\infty < t \leq -1$ and $1 \leq t < \infty$.

According to the results obtained thus far,

$$e^{-iH_a t} f_a = \sum_{i=1}^{3} f_i(t). \tag{2.6.34}$$

If W_{pq} is integrable and satisfies eq. (2.6.18),

$$\int_{|t|>1} ||V_{pq} f_i(t)|| dt = \text{const.} \int_{|t|>1} [I_i(t)]^{\frac{1}{2}} \frac{1}{t} dt < \infty \qquad (i = 1,2,3). \tag{2.6.35}$$

Now given eqs. (2.6.34) and (2.6.35), it follows from Minkowski's inequality that $||V_{pq}\exp(-iH_a t)f_a||$ is integrable over $-\infty < t \leq -1$ and $1 \leq t < \infty$. Combining this with our previous result for $-1 \leq t \leq 1$, we see that eq. (2.6.19) is fulfilled, as we wished to show.

In proving eq. (2.6.19), essential use has been made of the fact that the coordinates $\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1}$ were adapted to the particular function V_{pq} under discussion. Hence, this equation has thus far only been established for one single pair p,q. To extend it to all the interactions contained in V_a, we consider a term V_{rs} with $r,s \neq p,q$. By an orthogonal transformation among the coordinates $\boldsymbol{x}_{m+1},\ldots,\boldsymbol{x}_{2m-1}$, we go over to coordinates adapted to V_{rs}. This will entail a transformation among $\boldsymbol{k}_{m+1},\ldots,\boldsymbol{k}_{2m-1}$. However, if $\hat{f}(\boldsymbol{k}_{m+1},\ldots,\boldsymbol{k}_{2m-1})$ has bounded first-order partial derivatives with respect to the original coordinates, so it has with respect to the new ones. If it vanishes outside a bounded region before the transformation, then the same applies afterwards. Hence, after the transformation the methods developed for V_{pq} can be used for V_{rs} without alteration. From this it follows with Minkowski's inequality that $||V_a\exp(-iH_a t)f_a||$ is an integrable function of t. This function is also bounded, since f_a belongs to $\mathfrak{D}(H_0)$. Hence, if f_a belongs to $\mathfrak{F}_a$ and each function W_{pq} is integrable and satisfies eq. (2.6.18), the combination V_a, f_a is admissible.

2.6.5. Alternative restrictions on the interaction

If in eq. (2.6.18) μ is small, W_{pq} may be fairly singular, but at large distances it must tend to 0 rather fast. If μ is large, W_{pq} must not have serious singularities, except possibly at the origin. In this case W_{pq} need not tend to 0 very fast as x_h tends to ∞, a decrease faster than $x_h^{-\frac{7}{2}}$ being sufficient if $\mu = 2$. Our general arguments do not apply beyond $1 \leq \mu \leq 2$ because it is only in this interval that we can draw a conclusion of the form (2.6.24). However, the condition at infinity can be relaxed in a special case. For let us consider a function W_{pq} which is integrable and satisfies

$$W_{pq}(\boldsymbol{x}_h) < \text{const.}\, x_h^{-3-\eta} \qquad (x_h > R) \tag{2.6.36}$$

for some positive η. This function is conveniently written in the form

$$\left.\begin{aligned} & W_{pq}(\boldsymbol{x}_h) = W_<(\boldsymbol{x}_h) + W_>(\boldsymbol{x}_h), \\ & W_<(\boldsymbol{x}_h) = W_{pq}(\boldsymbol{x}_h), \qquad W_>(\boldsymbol{x}_h) = 0 \qquad (0 < x_h < R), \\ & W_<(\boldsymbol{x}_h) = 0, \qquad W_>(\boldsymbol{x}_h) = W_{pq}(\boldsymbol{x}_h) \qquad (x_h > R). \end{aligned}\right\} \tag{2.6.37}$$

Since $W_<$ is integrable and satisfies eq. (2.6.18) with $\mu = 1$, it can be discussed with the methods outlined above. In an obvious notation, it gives rise to functions $I_{i<}(t)$ $(i = 1,2,3)$ belonging to $\mathfrak{L}^{\nu/(\nu-1)}(t)$. Our previous methods also apply to the functions $I_{2>}$ and $I_{3>}$ associated with $W_>$. For these derive from the terms in eq. (2.6.21) proportional to 1 and $\boldsymbol{k}_h \cdot \text{grad}_h$, respectively. In analysing these terms, it was assumed only that W_{pq} is integrable.

To investigate the term proportional to $i\boldsymbol{k}_h \cdot \boldsymbol{x}_h$, we assume for simplicity that $\eta \leq 1/2$. There is no loss of generality in doing so. We drop the subscript h again and we use the variables $\boldsymbol{w}, v, \varphi, r$ described in eqs. (2.6.26) and (2.6.27). Expressing $\boldsymbol{x}$ in terms of polar coordinates x, χ, ψ in a frame of reference which has its polar axis in the direction of $\boldsymbol{w}$, we obtain

$$\left.\begin{aligned} (\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x} &= wx\cos\chi, \\ (\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x} &= vx\left[\frac{r}{vw}\cos\chi + \left(1-\frac{r^2}{v^2w^2}\right)^{\frac{1}{2}}\sin\chi(\cos\varphi\cos\psi + \sin\varphi\sin\psi)\right]. \end{aligned}\right\} \quad (2.6.38)$$

From this it follows that $(\boldsymbol{k}\cdot\boldsymbol{x})(\boldsymbol{k}'\cdot\boldsymbol{x})$ can be written as a linear combination of terms proportional to v^2x^2, vwx^2 and w^2x^2, respectively, the coefficients depending on $r/vw, \varphi, \chi, \psi$ and being expressible in terms of low-order spherical harmonics $Y_{lm}(\chi,\psi)$. Hence, in view of eq. (2.6.22),

$$\left.\begin{aligned} I_{1>}(t) &< \lim_{X\to\infty}\int_0^X dx\int_{-1}^{1} d\cos\chi\int_0^{2\pi} d\psi\int_{-K^2}^{K^2} dr\int_{w=\frac{|r|}{2K}}^{w=2K} d\boldsymbol{w}\int_{\frac{|r|}{w}}^{2K} dv\int_0^{2\pi} d\varphi A, \\ A &= x^{1-\eta}\exp[i(-rt+\boldsymbol{w}\cdot\boldsymbol{x})]\sum_{l=0}^{L}\sum_m Y_{lm}(\chi,\psi)w^{-1}v[(v^2+w^2)^2-4r^2]^{-1} \\ &\quad\times[v^2C_{lm1}(\boldsymbol{w},v,\varphi,r)+vwC_{lm2}(\boldsymbol{w},v,\varphi,r)+w^2C_{lm3}(\boldsymbol{w},v,\varphi,r)] \end{aligned}\right\} \quad (2.6.39)$$

with some set of bounded functions C. Integration with respect to $\cos\chi, \psi$, and φ yields

$$\left.\begin{aligned} I_{1>}(t) &< \lim_{X\to\infty}\int_0^X dx\int_{-K^2}^{K^2} dr\int_{w=\frac{|r|}{2K}}^{w=2K} d\boldsymbol{w}\int_{\frac{|r|}{w}}^{2K} dv x^{\frac{1}{2}-\eta}e^{-irt}\sum_{l=0}^{L}J_{l+\frac{1}{2}}(wx)w^{-\frac{3}{2}}v[(v^2+w^2)^2-4r^2]^{-1} \\ &\quad\times[v^2D_{l1}(\boldsymbol{w},v,r)+vwD_{l2}(\boldsymbol{w},v,r)+w^2D_{l3}(\boldsymbol{w},v,r)], \end{aligned}\right\} \quad (2.6.40)$$

the functions D being bounded. The integration with respect to x may be performed next, by Fubini's theorem. Also,

$$\int_0^X x^{\frac{1}{2}-\eta}J_{l+\frac{1}{2}}(wx)dx = w^{-\frac{3}{2}+\eta}G_X(w), \quad (2.6.41)$$

where $G_X(w)$ is bounded uniformly in X. Hence,

$$\lim_{X\to\infty}\int dr\int d\boldsymbol{w}\int dv\int_0^X dx = \int dr\int d\boldsymbol{w}\int dv\lim_{X\to\infty}\int_0^X dx, \quad (2.6.42)$$

by the theorem of dominated convergence. As a result,

$$\left.\begin{aligned} I_{1>}(t) < \int_{-K^2}^{K^2} dr \int_{w=\frac{|r|}{2K}}^{w=2K} d\boldsymbol{w} \int_{\frac{|r|}{w}}^{2K} dv\, e^{-irt} w^{-3+\eta} v[(v^2+w^2)^2 - 4r^2]^{-1} \\ \times [v^2 E_1(\boldsymbol{w},v,r) + vwE_2(\boldsymbol{w},v,r) + w^2 E_3(\boldsymbol{w},v,r)]. \end{aligned}\right\} \quad (2.6.43)$$

We now consider separately the terms E_1, E_2, and E_3. In an obvious notation, we write

$$I_{1>}(t) < I_{11>}(t) + I_{12>}(t) + I_{13>}(t). \quad (2.6.44)$$

Owing to eqs. (2.6.32) and (2.6.33),

$$I_{13>}(t) = \int_{-K^2}^{K^2} e^{-irt} \log|r| F_3(r) dr, \quad (2.6.45)$$

where F_3 is a bounded function. Hence $I_{13>}(t)$ belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)(1 < \nu < 2)$, by previous arguments.

For the term $I_{11>}$ we consider

$$\left.\begin{aligned} & \int_{\frac{|r|}{2K}}^{2K} dw \int_{\frac{|r|}{w}}^{2K} dv\, w^{-1+\eta} v^3 [(v^2+w^2)^2 - 4r^2]^{-1} \\ & = \frac{1}{8r} \int_{\frac{|r|}{2K}}^{2K} w^{1+\eta} \log \frac{(4K^2+w^2+2r)(w^2-r)^2}{(4K^2+w^2-2r)(w^2+r)^2} dw \\ & + \tfrac{1}{4} \int_{\frac{|r|}{2K}}^{2K} w^{-1+\eta} \log \frac{(4K^2+w^2+2r)(4K^2+w^2-2r)w^4}{(w^2+r)^2(w^2-r)^2} dw. \end{aligned}\right\} \quad (2.6.46)$$

Here the first term on the right equals $\log|r|$ times a bounded function of r, again by eqs. (2.6.32) and (2.6.33). Since $0 < \eta \leq 1/2$ by assumption, $w^{-1+\eta} \leq (|r|/2K)^{-1+\eta}$. Also, the logarithm in the second term on the right is integrable. Hence, the second term on the right is equal to $r^{-1+\eta}$ times a bounded function of r. Summarizing,

$$I_{11>}(t) = \int_{-K^2}^{K^2} e^{-irt} r^{-1+\eta} F_1(r) dr, \quad (2.6.47)$$

with some bounded function F_1. This shows that, if ν satisfies $1 < \nu < (1-\eta)^{-1} \leq 2$, the function $I_{11>}(t)$ is the Fourier transform of a function in $\mathfrak{L}^{\nu}(r)$. Hence $I_{11>}(t)$ belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)$.

It is now obvious that $I_{12>}(t)$ also belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)$, for some suitable set of numbers ν. For, given eqs. (2.6.45) and (2.6.47), it follows from Schwarz's inequality that

$$I_{12>}(t) = \int_{-K^2}^{K^2} e^{-irt}(r^{-1+\eta}\log|r|)^{\frac{1}{2}} F_2(r)dr, \tag{2.6.48}$$

and the argument can be completed as before. Summarizing, each function $I_{1i>}(t)$ $(i = 1,2,3)$ belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)$. Hence so does $I_{1>}(t)$, by eq. (2.6.44).

Since W_{pq} is the sum of $W_<$ and $W_>$, the function $I_i(t)$ is the sum of $I_{i<}(t)$ and $I_{i>}(t)$. Hence $I_i(t)$ belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)$. Equation (2.6.35) is therefore satisfied also in the present case. From this it follows as before that $||V_{pq}\exp(-iH_a t)f_a||$ is integrable over $-\infty < t \leq -1$ and $1 \leq t < \infty$, next that the same applies to $||V_a\exp(-iH_a t)f_a||$. Since either norm is a bounded function of t, the final conclusion is that the quantity $||V_a\exp(-iH_a t)f_a||$ is bounded and belongs to $\mathfrak{L}(t)$. Hence the combination V_a, f_a is admissible, as we wished to show.

According to the results obtained thus far, V_a, f_a is admissible (i.e. in eqs. (2.6.1) and (2.6.4) the limits with respect to ε and δ may be interchanged) whenever f_a belongs to $\mathfrak{F}_a$ and each W_{pq} is an integrable function satisfying either eq. (2.6.18) or eq. (2.6.36). This is due to certain functions $I(t)$ belonging to $\mathfrak{L}^{\nu/(\nu-1)}(t)$. Now the functions $I(t)$ depend linearly on W_{pq}, by eq. (2.6.22). Hence, if f_a belongs to $\mathfrak{F}_a$ and each W_{pq} is integrable, for V_a, f_a to be admissible it is already sufficient if W_{pq} is some linear combination of functions satisfying eq. (2.6.18) or (2.6.36), possibly with different exponents μ. This is the most general result we have obtained in this connection.

2.6.6. Examples of admissible interactions

It does not seem easy to translate the conditions on W_{pq} into conditions on V_{pq}, unless something is known about the asymptotic behaviour of the functions $\varphi_{(j)}(\boldsymbol{x}_j)$. We recall here that in eq. (2.6.16) the quantity $\sum_r d_r \boldsymbol{x}_{j(p),r}$ stands for the distance between particle p and the centre of mass of fragment $j(p)$. Now given this interpretation, it will be shown in a separate paper that, if all the interactions V_{ij} within the fragment $j(p)$ are square-integrable and the eigenvalue $\lambda_{(j(p))}$ does not belong to the continuous spectrum of the Hamiltonian $H_{(j(p))}(\boldsymbol{x}_{j(p)})$,

$$N_{p\alpha} \equiv \int |(\sum_r d_r \boldsymbol{x}_{j(p),r})^{\alpha} \varphi_{(j(p))}(\boldsymbol{x}_{j(p)})|^2 d\boldsymbol{x}_{j(p)} < \infty \tag{2.6.49}$$

for every $\alpha \geq 0$. This result ensues from a study of the analytic properties of scattering amplitudes, which is beyond the scope of the present paper. However, let us assume that eq. (2.6.49) holds true. Let us also assume that

$$U_\mu \equiv \int [V_{pq}(\boldsymbol{X})X]^{2\mu} d^3\boldsymbol{X} < \infty, \qquad V_\mu \equiv \int [V_{pq}(\boldsymbol{X})]^{2\mu} d^3\boldsymbol{X} < \infty, \tag{2.6.50}$$

with $\mu = 1$. The inequality

$$x_h^\beta \leq (3/c)^\beta \left(|\boldsymbol{y}_p + \boldsymbol{y}_q + c\boldsymbol{x}_h|^\beta + y_p^\beta + y_q^\beta\right) \qquad (\beta \geq 0), \tag{2.6.51}$$

applied with $\beta = 2$, then yields

$$\int W_{pq}(\boldsymbol{x}_h) x_h^2 \, d\boldsymbol{x}_h < \text{const.}(U_1 N_{p0} N_{q0} + V_1 N_{p1} N_{q0} + V_1 N_{p0} N_{q1}) \tag{2.6.52}$$

(cf. eqs. (2.6.16) and (2.6.17)). Hence eq. (2.6.18) is satisfied with $\mu = 1$. Similarly, if eq. (2.6.50) is true for $\mu = 2$, so is eq. (2.6.18). Both for $\mu = 1$ and for $\mu = 2$, it also follows from eq. (2.6.50) that V_{pq} is square-integrable, hence that W_{pq} is integrable. This shows that, if f_a belongs to $\mathfrak{F}_a$, if eq. (2.6.49) holds true for $\alpha = 1$, and if each V_{pq} satisfies eq. (2.6.50) with either $\mu = 1$ or $\mu = 2$, the combination V_a, f_a is admissible.

Again, let us consider the case that each V_{pq} is square-integrable and such that

$$|V_{pq}(\boldsymbol{X})| < \text{const.} X^{-\frac{3}{2} - \frac{1}{2}\eta} \qquad (X > R) \tag{2.6.53}$$

for some positive η. In this case it is convenient to write

$$\left.\begin{aligned} V_{pq}(\boldsymbol{X}) &= V_<(\boldsymbol{X}) + V_>(\boldsymbol{X}), \\ [V_{pq}(\boldsymbol{X})]^2 &= [V_<(\boldsymbol{X})]^2 + [V_>(\boldsymbol{X})]^2, \end{aligned}\right\} \tag{2.6.54}$$

the functions $V_<$ and $V_>$ being defined as in eq. (2.6.37). Since $V_<$ satisfies eq. (2.6.50) with $\mu = 1$, it yields a function W with $\mu = 1$. Also, $V_>$ yields a function W of the form (2.6.36). For if in eq. (2.6.51) we take $\beta = 3 + \eta$, we obtain

$$\left.\begin{aligned} &\int \Big| V_>\big(\sum_r d_r \boldsymbol{x}_{j(p),r} + \sum_s d_s \boldsymbol{x}_{j(q),s} + c\boldsymbol{x}_h\big) \varphi_{(j(p))}(\boldsymbol{x}_{j(p)}) \varphi_{(j(q))}(\boldsymbol{x}_{j(q)}) \Big|^2 d\boldsymbol{x}_{j(p)} d\boldsymbol{x}_{j(q)} \\ &< \text{const.} x_h^{-3-\eta} \left(N_{p0} N_{q0} + N_{p,\frac{3}{2}+\frac{1}{2}\eta} N_{q0} + N_{p0} N_{q,\frac{3}{2}+\frac{1}{2}\eta} \right) \qquad (x_h > R). \end{aligned}\right\} \tag{2.6.55}$$

Hence, if eq. (2.6.49) holds true for $\alpha = (3 + \eta)/2$ and if each V_{pq} is a square-integrable function satisfying eq. (2.6.53), each W_{pq} is the sum of two suitable functions. If f_a belongs to $\mathfrak{F}_a$, the combination V_a, f_a is admissible.

2.6.7. The scattering operators in momentum space

If f_a belongs to $\mathfrak{F}_a$ and each function W_{pq} satisfies the sufficient conditions summarized at the end of section 2.6.5, in eqs. (2.6.1) and (2.6.4) the limits with respect to ε and δ may be interchanged. Going over to the Fourier transform of f_a, we can then perform the limit with respect to δ. This yields an expression for the operator S_{ba} in momentum space which is the subject of the present section.

In discussing the quantity $(g_b, S_{ba} f_a)$, we obviously have to assume that V_b is such that the operator Ω_{b-} exists. Also, g_b must belong to $\mathfrak{C}_b$. Thus far no further restrictions were imposed upon g_b. From now on we assume explicitly that g_b is of the product-form (2.3.4).

In applying the results of the last few sections, it is convenient to start with a function $E_b(K_b^2)g_b$ the Fourier transform of which vanishes outside a region characterized by K_b, and to perform the limit with respect to K_b as the last step. Obviously this is always permitted. In this way we are led to consider

$$\lim_{K_b\to\infty}\ \lim_{\varepsilon\to 0}\ \lim_{\delta\to 0}. \tag{2.6.56}$$

Since, under the present assumptions, the quantity $||V_a\exp(-iH_at)f_a||$ is bounded and integrable, the integral

$$\int_{-\infty}^{\infty} e^{-\delta|t|}(E_b(K_b^2)g_b, e^{iH_bt}V_ae^{-iH_at}f_a)dt \tag{2.6.57}$$

tends to a limit as δ tends to 0. Hence, in eq. (2.6.4) the quantities C and D each have a limit with respect to δ.

For the transition to Fourier transforms it is convenient to use the operators $P_a(X_a)$ and $P_b(X_b)$ defined in eq. (2.5.25). Replacing in eq. (2.6.4) $R_a(u\pm i\delta)$ by $P_a(X_a)R_a(u\pm i\delta)$ and $R_b(u\pm i\zeta)$ by $R_b(u\pm i\zeta)P_b(X_b)$, we have to evaluate

$$\lim_{K_b\to\infty}\ \lim_{\varepsilon\to 0}\ \lim_{\delta\to 0}\{\lim_{X_b\to\infty}\ \lim_{X_a\to\infty}\}\lim_{\zeta\to 0}, \tag{2.6.58}$$

by analogy with eq. (2.5.27). We now show first that in eq. (2.6.4)

$$\lim_{\delta\to 0}\{\lim_{X_b\to\infty}\ \lim_{X_a\to\infty}\}\lim_{\zeta\to 0} = \{\lim_{X_b\to\infty}\ \lim_{X_a\to\infty}\}\lim_{\delta\to 0}\ \lim_{\zeta\to 0}. \tag{2.6.59}$$

This is most easily proved with the help of eq. (2.6.1). For we have

$$\left.\begin{aligned} e^{-\delta|t|}\,&|(E_b(K_b^2)g_b, e^{iH_b(s+t)}P_b(X_b)V_be^{-iHs}V_aP_a(X_a)e^{-iH_at}f_a)|\\ &< ||V_be^{-iH_b(s+t)}E_b(K_b^2)g_b||\;||V_ae^{-iH_at}f_a||\\ &< (\alpha||H_0E_b(K_b^2)g_b||+\beta||E_b(K_b^2)g_b||)||V_ae^{-iH_at}f_a||,\end{aligned}\right\} \tag{2.6.60}$$

the second inequality following from eq. (2.1.4). According to eq. (2.6.60), in the last integral in eq. (2.6.1) the integrand does not exceed an integrable function of t which does not depend on X_a, X_b, and δ. Hence, if $\varepsilon>0$,

$$\left.\begin{aligned} \lim_{\delta\to 0}\int dt\int ds\{\lim_{X_b\to\infty}\ \lim_{X_a\to\infty}\} &= \int dt\int ds\{\lim_{\delta\to 0}\ \lim_{X_b\to\infty}\ \lim_{X_a\to\infty}\}\\ &= \{\lim_{\delta\to 0}\ \lim_{X_b\to\infty}\ \lim_{X_a\to\infty}\}\int dt\int ds,\end{aligned}\right\} \tag{2.6.61}$$

by the theorem of bounded convergence. And similarly for the first integral on the right in eq. (2.6.1). If from eq. (2.6.1) we now deduce eq. (2.6.4), the limit-property (2.6.59) follows as an obvious consequence.

The limit with respect to ζ can now be performed with the methods of section 2.5.3. In writing it down, we use the notation in terms of $\boldsymbol{x}_a', \boldsymbol{x}_a$ introduced in section 2.5.3. Since $\boldsymbol{x}_a'$ stands for the internal coordinates, $\boldsymbol{x}_a$ for the relative coordinates of the m_a fragments into which the system is split when it is in channel a, it is appropriate to express f_a in terms of $\boldsymbol{x}_a', \boldsymbol{x}_a$. For convenience we express g_b in terms of $\boldsymbol{x}_b', \boldsymbol{x}_b$, this being a set of coordinates which is related to $\boldsymbol{x}_a', \boldsymbol{x}_a$ through an orthogonal transformation.

In the new notation, taking the limit with respect to ζ yields

$$\left.\begin{aligned}(g_b, S_{ba} f_a) &= \delta_{ba}(g_b, f_a) + (2\pi)^{-\frac{3}{2}m_b+\frac{3}{2}} \lim_{K_b\to\infty} \lim_{\varepsilon\to 0} \{ \lim_{X_b\to\infty} \lim_{X_a\to\infty} \} \lim_{\delta\to 0} \int_{k_b\leq K_b} I d\boldsymbol{k}_b, \\ I &= \bar{g}(\boldsymbol{k}_b)(P_b(X_b)\varphi_b e^{i\boldsymbol{k}_b\cdot\boldsymbol{x}_b}, [-1 + V_b R(k_b^2+\lambda_b+i\varepsilon)] V_a P_a(X_a) \\ &\quad \times [R_a(k_b^2+\lambda_b+i\delta) - R_a(k_b^2+\lambda_b-i\delta)]\varphi_a f).\end{aligned}\right\} \quad (2.6.62)$$

In going over to the Fourier transform of f, it is now a question of

$$\lim_{\delta\to 0} \int_{k_b\leq K_b} d\boldsymbol{k}_b \int d\boldsymbol{k}_a \bar{g}(\boldsymbol{k}_b) M(\boldsymbol{k}_b, \boldsymbol{k}_a) \frac{\delta}{(k_b^2+\lambda_b-k_a^2-\lambda_a)^2+\delta^2} \hat{f}(\boldsymbol{k}_a), \quad (2.6.63)$$

where M can be found from eq. (2.6.62). Since $P_b(X_b)\varphi_b$ and $V_a P_a(X_a)\varphi_a$ belong to $\mathfrak{L}^2$ and $V_b R(k_b^2+\lambda_b+i\varepsilon)$ is a bounded operator, the function M is bounded. Hence, since f belongs to $\mathfrak{F}$ by assumption, the integral $\int d\boldsymbol{k}_a$ does not exceed an integrable function of $\boldsymbol{k}_b$, uniformly in δ. As a result

$$\lim_{\delta\to 0} \int d\boldsymbol{k}_b \int d\boldsymbol{k}_a = \int d\boldsymbol{k}_b \lim_{\delta\to 0} \int d\boldsymbol{k}_a. \quad (2.6.64)$$

Performing the limit with respect to δ now shows that in the final result there will be contributions only from the region where

$$k_a^2 + \lambda_a = k_b^2 + \lambda_b = E. \quad (2.6.65)$$

This brings out the conservation of energy during the scattering process. Since k_b is thereby restricted automatically, the limit with respect to K_b may be omitted from eq. (2.6.62). Expressing $\boldsymbol{k}_i$ ($i = a,b$) in terms of k_i and various polar angles ω_{k_i}, with

$$\int d\boldsymbol{k}_i = \int k_i^{3m_i-4} dk_i d\omega_{k_i} \qquad (i = a,b), \quad (2.6.66)$$

we thus obtain

$$\left.\begin{aligned}(g_b, S_{ba}f_a) &= \delta_{ba}(g_b, f_a)\\ &+ \tfrac{1}{4}i(2\pi)^{-\frac{3}{2}(m_b+m_a)+4}\lim_{\varepsilon\to 0}\Big\{\lim_{X_b\to\infty}\lim_{X_a\to\infty}\Big\}\int dE\iint d\omega_{k_b}d\omega_{k_a}k_b^{3m_b-5}k_a^{3m_a-5}\bar{\hat{g}}(\boldsymbol{k}_b)F(\boldsymbol{k}_b,\boldsymbol{k}_a;\varepsilon;X_b,X_a)\hat{f}(\boldsymbol{k}_a),\\ F(\boldsymbol{k}_b,\boldsymbol{k}_a;\varepsilon;X_b,X_a) &= (P_b(X_b)\varphi_b e^{i\boldsymbol{k}_b\cdot\boldsymbol{x}_b}, [-1+V_bR(E+i\varepsilon)]V_aP_a(X_a)\varphi_a e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a}).\end{aligned}\right\} \quad (2.6.67)$$

It follows from our discussion of eq. (2.2.11) that $S_{ba}f_a$ belongs to $\mathfrak{C}_b$. Let us now assume that the two-body interactions V_{ij} contained in V_b satisfy eq. (2.4.25). Then all functions in $\mathfrak{C}_b$ are of the form $\varphi_b(\boldsymbol{x}_b')h(\boldsymbol{x}_b)$, by section 2.4.3. In particular,

$$\left.\begin{aligned}(S_{ba}-\delta_{ba})f_a &= \varphi_b(\boldsymbol{x}_b')h(\boldsymbol{x}_b),\\ (g_b, [S_{ba}-\delta_{ba}]f_a) &= \int \bar{\hat{g}}(\boldsymbol{k}_b)\hat{h}(\boldsymbol{k}_b)d\boldsymbol{k}_b,\end{aligned}\right\} \quad (2.6.68)$$

with some function h in $\mathfrak{L}^2$.

If in eq. (2.6.67) the integration with respect to ω_{k_a} is performed first, the remaining integral has an integrand which essentially depends only on k_b and ω_{k_b}, the quantities E and k_a being determined by k_b according to eq. (2.6.65). Hence, combining eqs. (2.6.67) and (2.6.68), we obtain a relation of the form

$$\int \bar{\hat{g}}(\boldsymbol{k}_b)\hat{h}(\boldsymbol{k}_b)d\boldsymbol{k}_b = \lim_N \int \bar{\hat{g}}(\boldsymbol{k}_b)\hat{h}_N(\boldsymbol{k}_b)d\boldsymbol{k}_b, \quad (2.6.69)$$

where N stands for ε, X_b, X_a. Taking in particular $g = h$ and applying eq. (2.6.69) once more yields

$$\int |\hat{h}(\boldsymbol{k}_b)|^2 d\boldsymbol{k}_b = \lim_N \lim_{N'} \int \bar{\hat{h}}_{N'}(\boldsymbol{k}_b)\hat{h}_N(\boldsymbol{k}_b)d\boldsymbol{k}_b. \quad (2.6.70)$$

The probability of scattering into channel b thus takes the form

$$\left.\begin{aligned}&||(S_{ba}-\delta_{ba})f_a||^2\\ &= \tfrac{1}{4}(2\pi)^{-3(m_b+m_a)+8}\lim_{\varepsilon\to 0}\Big\{\lim_{X_b\to\infty}\lim_{X_a\to\infty}\Big\}\lim_{\varepsilon'\to 0}\Big\{\lim_{X_b'\to\infty}\lim_{X_a'\to\infty}\Big\}\int \bar{A}(\boldsymbol{k}_b;\varepsilon';X_b',X_a')A(\boldsymbol{k}_b;\varepsilon;X_b,X_a)d\boldsymbol{k}_b,\\ &A(\boldsymbol{k}_b;\varepsilon;X_b,X_a) = \int k_a^{3m_a-5}F(\boldsymbol{k}_b,\boldsymbol{k}_a;\varepsilon;X_b,X_a)\hat{f}(\boldsymbol{k}_a)d\omega_{k_a}.\end{aligned}\right\} \quad (2.6.71)$$

2.6.8. Discussion

The results of the previous section are subject to the assumption that f_a belongs to $\mathfrak{F}_a$, and that V_a satisfies the sufficient conditions discussed in sections 2.6.4 and 2.6.5. These restrictions are imposed to guarantee that in eq. (2.6.67) the limits with respect to ε and X_b, X_a do in fact exist. In eq. (2.6.1) we have introduced the repeated limit with respect to ε and δ to replace the limits with respect to the time variables s and t. These entered the problem because we have considered transitions from an initial state in the distant past to a final state in the remote future. The condition that there should be a limit as ε tends to 0 is very closely related to the condition that in a scattering experiment the scattered fragments should become free sufficiently rapidly as the time t goes to ∞. This condition is fulfilled if the interaction between any two fragments tends to 0 sufficiently rapidly as their distance increases. Roughly speaking, we may therefore say that the behaviour of the resolvent in the neighbourhood of the continuous spectrum reflects the asymptotic properties of scattered wave-packets at very large times. These in turn depend on the interaction, and in particular on its behaviour at large distances, according to the foregoing analysis.

By taking ε sufficiently small but different from 0, a good approximation can be obtained to the limit in eq. (2.6.67). Doing so may have advantages owing to the fact that off the real axis the resolvent can be studied much more easily than in the continuous spectrum. As a matter of fact, it was shown in part I of the present investigation that, if all the two-body interactions in the system are square-integrable, there is a systematic procedure for constructing the resolvent for non-real energies. This method breaks down in the continuous spectrum. From the general theory of resolvents and spectra one gets the impression that this is a fundamental difficulty which cannot be solved within the framework of a Hilbert-space formalism. In view of the correspondence between small values of ε and large times, we may say that taking a non-vanishing ε is analogous to extending a scattering experiment over a finite time only. The analogy should not be taken too literally, however, the precise relationship between non-real energies and finite times being fairly complicated.

In eq. (2.6.67) there is also a limit with respect to X_b, X_a. If X_a is finite, the interaction V_a is distorted in so far as it takes place outside a hypersphere of radius X_a in $\boldsymbol{x}_a$-space. Hence, taking X_a large but finite may yield a good approximation, provided the properties of the system are such that, if any two fragments are far apart, all the fragments are effectively free. This is again a condition which refers to the interaction between the fragments decreasing sufficiently rapidly. Accordingly, once it is known that there is a limit as ε tends to 0, the limit with respect to X_b, X_a does not present further difficulties. This we saw in the proof of eq. (2.6.59). Letting X_b, X_a remain finite corresponds to performing a scattering experiment in a finite region of space.

It is a crucial point that in eq. (2.6.67) the integration must be performed before the limits are taken. This equation thus contains a statement concerning a sequence of linear functionals. It is essentially the kind of relation one studies in the theory of distributions, the function $\bar{\hat{g}}\hat{f}$ playing the part of the test-function. From this point of view it is not surprising that $\hat{f}$ must satisfy certain conditions of smoothness. In this connection it may be remarked that, since in measurements there are always errors, one cannot distinguish experimentally between a wave-function f_1 and a wave-function f_2 which is almost equal to f_1 in the sense that $\|f_1 - f_2\| < \eta$, where η is a small positive number determined by the accuracy of the experiment in question. Hence, in practice no loss of generality is involved if the initial state of a scattering process is described in terms of a set of functions which is dense in $\mathfrak{C}_a$.

2.7. The scattering of two fragments

2.7.1. Restrictions on the interaction and on the relative motion

If in channel a the system is split into not more than two fragments, it is convenient to develop the function $\hat{f}(\boldsymbol{k}_a)$ in terms of spherical harmonics, $\boldsymbol{k}_a$ now being a three-dimensional vector. In many cases of practical interest we have

$$\hat{f}(\boldsymbol{k}_a) = \sum_{l=0}^{L} \sum_{m} \hat{f}_{lm}(k_a) Y_{lm}(\omega_{k_a}), \tag{2.7.1}$$

where L is finite. In any case the function $\hat{f}$ can be approximated in mean square by a sum of the form (2.7.1).

If $\hat{f}$ is of the form (2.7.1) and each $\hat{f}_{lm}$ is suitably restricted, the combination $V_{a'}f_a$ is admissible under conditions on V_a which are much less stringent than the ones imposed in sections 2.6.4 and 2.6.5. The present section is devoted to this point.

We assume for the time being that $\hat{f}_{lm}$ belongs to the set $\hat{\mathfrak{G}}_{lm}$ consisting of all functions defined in $0 \leq k_a < \infty$ which have bounded derivatives of the first order and vanish outside bounded intervals. If each $\hat{f}_{lm}$ belongs to $\hat{\mathfrak{G}}_{lm}$ and $\hat{f}$ is of the form (2.7.1), we say that $\hat{f}$ belongs to $\hat{\mathfrak{G}}$. The set of functions f which are Fourier transforms of functions in $\hat{\mathfrak{G}}$ is denoted by $\mathfrak{G}$. The set of functions f_a which are equal to $\varphi_a(\boldsymbol{x}'_a)$ times a function $f(\boldsymbol{x}_a)$ in $\mathfrak{G}$ is denoted by $\mathfrak{G}_a$.

It is not difficult to see that $\mathfrak{G}_a$ is dense in the set of functions (2.3.4). To show that this is so, we merely have to prove that, given a positive ξ and a function $\hat{f}_{lm}(k_a)$ such that

$$\int_0^{\infty} |\hat{f}_{lm}(k_a)|^2 k_a^2\, dk_a < \infty, \tag{2.7.2}$$

there is a function $\hat{g}(k_a)$ in $\hat{\mathfrak{G}}_{lm}$ such that

$$\int_0^{\infty} |\hat{f}_{lm}(k_a) - \hat{g}(k_a)|^2 k_a^2 dk_a < \xi. \tag{2.7.3}$$

Now if eq. (2.7.2) holds true, $\hat{f}_{lm}(k_a)k_a$ can be approximated in mean square by a function which is equal to $\hat{f}_{lm}(k_a)k_a$ in some interval $0 < 3\eta \leq k_a < K-2\eta$ and vanishes everywhere else. This function can be approximated in mean square by a continuous function which vanishes outside the interval $2\eta \leq k_a \leq K-\eta$ (McShane (25) section 42.4s). Next, the continuous function can be approximated uniformly by a continuously differentiable function which vanishes unless $\eta \leq k_a \leq K$ (Schwartz (26) ch. I, theorem 1). Let us denote the last function by $\hat{h}(k_a)$. Since $\hat{h}(k_a)$ vanishes in the neighbourhood of the origin, we can consider the function $\hat{g}(k_a) = \hat{h}(k_a)/k_a$. This clearly belongs to $\hat{\mathfrak{G}}_{lm}$. By the foregoing argument, it can be chosen in such a way that, in the sense of eq. (2.7.3), it approximates $\hat{f}_{lm}(k_a)$ as close as we please.

Hence in this sense $\hat{\mathfrak{G}}_{lm}$ is dense in the set of functions $\hat{f}_{lm}(k_a)$ which satisfy eq. (2.7.2). From this it follows that $\mathfrak{G}_a$ is dense in the set of functions (2.3.4).

Since in the special case of the present section $\boldsymbol{x}_a$ is proportional to the distance between the centres of mass of the two scattered fragments, it is automatically equal to the coordinate $\boldsymbol{x}_h$ used in section 2.6.3. Expressing $\boldsymbol{x}_a$ in terms of polar coordinates x_a, ω_{x_a}, we define

$$Q_{pq}(x_a) = \int W_{pq}(\boldsymbol{x}_a) d\omega_{x_a}. \tag{2.7.4}$$

Instead of eq. (2.6.18) we now consider the inequality

$$\int [Q_{pq}(x_a) x_a^2]^{\mu} dx_a < \infty, \tag{2.7.5}$$

μ being restricted again to $1 \leq \mu \leq 2$. If eq. (2.7.5) holds true, it follows from Hölder's inequality that

$$\int Q_{pq}(x_a)(1+x_a)^{-1+\zeta} x_a^2 dx_a < \infty \tag{2.7.6}$$

whenever $\zeta < 1/\mu$. Hence the wave-operator Ω_{a+} certainly exists, the sufficient condition (2.4.18) being fulfilled.

2.7.2. The convergence problem

We proceed to show that, if f_a belongs to $\mathfrak{G}_a$ and each two-body interaction V_{pq} contained in V_a is such that the respective equation (2.7.5) is satisfied, the combination V_a, f_a is admissible. The argument runs in many ways parallel to section 2.6.4, the crucial point being again the proof that $||V_{pq}\exp(-iH_a t)f_a||$ belongs to $\mathfrak{L}(t)$.

Assuming for simplicity that the sum in eq. (2.7.1) consists of only one term and dropping the subscript a, we write

$$\left.\begin{aligned} \exp[i\Delta(\boldsymbol{x})t]f(\boldsymbol{x}) &= (2\pi)^{-\frac{3}{2}} \int \exp[i(-k^2 t + \boldsymbol{k}\cdot\boldsymbol{x})]\hat{f}_{lm}(k) Y_{lm}(\omega_k) d\boldsymbol{k} \\ &= i^l Y_{lm}(\omega_x) \int \exp(-ik^2 t) \frac{1}{\sqrt{kx}} J_{l+\frac{1}{2}}(kx) \hat{f}_{lm}(k) k^2 dk \\ = \frac{i^{l-1}}{2t} Y_{lm}(\omega_x) \int \exp(-ik^2 t) &\left[\frac{l+1}{\sqrt{kx}} J_{l+\frac{1}{2}}(kx) - \sqrt{kx} J_{l+\frac{3}{2}}(kx) + \frac{k}{\sqrt{kx}} J_{l+\frac{1}{2}} \frac{d}{dk}\right] \hat{f}_{lm}(k) dk, \end{aligned}\right\} \tag{2.7.7}$$

the last member following upon integration by parts. Now

$$\sqrt{kx} J_{l+\frac{3}{2}}(kx) = 2(2\pi)^{-\frac{1}{2}} \sin(kx - \tfrac{1}{2} l\pi - \tfrac{1}{2}\pi) + R_{l+\frac{3}{2}}(kx), \tag{2.7.8}$$

where the remainder $R_{l+\frac{3}{2}}(kx)$ does not exceed a constant times $(1+kx)^{-1}$. Also, $(kx)^{-\frac{1}{2}} J_{l+\frac{1}{2}}(kx)$ does not exceed a constant times $(1+kx)^{-1}$. Hence, if we write

$$\left.\begin{aligned}&\exp[i\Delta(\boldsymbol{x})t]f(\boldsymbol{x})\\ &= \frac{i^{l-1}}{t}Y_{lm}(\omega_x)\int\exp(-ik^2t)[-(2\pi)^{-\frac{1}{2}}\sin(kx-\tfrac{1}{2}l\pi-\tfrac{1}{2}\pi)\hat{f}_{lm}(k)+S(k,x)]dk,\end{aligned}\right\} \quad (2.7.9)$$

the function $S(k,x)$ does not exceed a constant times $(1+kx)^{-1}$. It vanishes if k is larger than some finite K, in virtue of our assumption that $\hat{f}_{lm}$ belongs to $\hat{\mathfrak{G}}_{lm}$.

Let us now consider the relation

$$\left.\begin{aligned}I_1(t) &\equiv \int W_{pq}(\boldsymbol{x})|Y_{lm}(\omega_x)\int\exp(-ik^2t)\sin(kx-\tfrac{1}{2}l\pi-\tfrac{1}{2}\pi)\hat{f}_{lm}(k)dk|^2d\boldsymbol{x}\\ &< \text{const.}\int Q_{pq}(x)x^2dx\int\int\exp[-i(k^2-k'^2)t]\\ &\times[(-1)^l\cos(kx+k'x)+\cos(kx-k'x)]\hat{f}_{lm}(k)\bar{\hat{f}}_{lm}(k')dkdk'.\end{aligned}\right\} \quad (2.7.10)$$

If eq. (2.7.5) holds true for some μ in the interval $1<\mu\leq 2$, the quantities

$$Z_{X\pm}(k\pm k') = (2\pi)^{-\frac{1}{2}}\int_0^X Q_{pq}(x)x^2\cos(kx\pm k'x)dx \quad (2.7.11)$$

satisfy

$$\left[(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}|Z_{X\pm}(k\pm k')|^{\mu/(\mu-1)}d(k\pm k')\right]^{(\mu-1)/\mu} < \left[(2\pi)^{-\frac{1}{2}}\int_0^X[Q_{pq}(x)x^2]^\mu dx\right]^{1/\mu} \quad (2.7.12)$$

(cf. eq. (2.6.24)). If in eq. (2.7.5) $\mu=1, Z_{X\pm}$ is bounded uniformly in X. Hence in the third member of eq. (2.7.10)

$$\lim_{X\to\infty}\int_0^X dx\int\int dkdk' = \int\int dkdk'\lim_{X\to\infty}\int_0^X dx. \quad (2.7.13)$$

Denoting $Z_{\infty\pm}$ by $Z_\pm$, we thus obtain

$$I_1(t) < \int\int[(-1)^lZ_+(k+k')+Z_-(k-k')]\exp[-i(k^2-k'^2)t]F(k,k')dkdk', \quad (2.7.14)$$

where F is bounded and vanishes outside a bounded region.

In an obvious notation we now write

$$I_1(t) < I_{1+}(t)+I_{1-}(t). \quad (2.7.15)$$

In terms of the variables

$$k^2-k'^2 = r, \qquad k+k' = v \quad (2.7.16)$$

the function $I_{1+}(t)$ takes the form

$$I_{1+}(t) = \int_{-K^2}^{K^2} dr \int_{\substack{|r|\\K}}^{2K} dv e^{-irt} Z_+(v) v^{-1} G(v,r), \tag{2.7.17}$$

with some bounded function G. Hence, if $\mu = 1$, $I_{1+}(t)$ is the Fourier transform of $\log|r|$ times a bounded function of r. If $1 < \mu \leq 2$, we have the inequality

$$\int_{\substack{|r|\\K}}^{2K} |Z_+(v)| v^{-1} dv \leq \left[\int |Z_+(v)|^{\mu/(\mu-1)} dv\right]^{(\mu-1)/\mu} \left[\int_{\substack{|r|\\K}}^{2K} v^{-\mu} dv\right]^{1/\mu} \leq \text{const.} |r|^{(1-\mu)/\mu}. \tag{2.7.18}$$

In either case $I_{1+}(t)$ is the Fourier transform of a function in $\mathfrak{L}^{\nu}(r)$ $(1 < \nu < 2)$. Hence $I_{1+}(t)$ belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)$. By a similar argument $I_{1-}(t)$ belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)$. Hence so does $I_1(t)$.

We now turn to the term $S(k,x)$ in eq. (2.7.9). This yields a function $I_2(t)$ such that

$$I_2(t) = \iint \exp[-i(k^2 - k'^2)t](kk')^{(1-\mu)/2\mu} H(k,k') dk dk', \tag{2.7.19}$$

where H vanishes outside a bounded region and satisfies

$$\left.\begin{aligned} |H(k,k')| &\leq (kk')^{(\mu-1)/2\mu} \int W_{pq}(\boldsymbol{x}) |Y_{lm}(\omega_x)|^2 |S(k,x) S(k',x)| d\boldsymbol{x} \\ &< \text{const.} (kk')^{(\mu-1)/2\mu} \int Q_{pq}(x) x^2 (1+kx)^{-1} (1+k'x)^{-1} dx \\ &< \text{const.} (kk')^{(\mu-1)/2\mu} \left[\int [Q_{pq}(x) x^2]^{\mu} dx\right]^{1/\mu} \left[\int [(1+kx)(1+k'x)]^{\mu/(1-\mu)} dx\right]^{(\mu-1)/\mu} < \text{const.} \end{aligned}\right\} \tag{2.7.20}$$

In terms of the variables r and v we have

$$I_2(t) = \int_{-K^2}^{K^2} dr \int_{\substack{|r|\\K}}^{2K} dv e^{-irt} \left(v^2 - \frac{r^2}{v^2}\right)^{(1-\mu)/2\mu} v^{-1} J(v,r), \tag{2.7.21}$$

with some bounded function J. If $1 < \mu \leq 2$, the substitution $v^4 = r^2 q$ yields

$$\int_{\substack{|r|\\K}}^{2K} \left|v^2 - \frac{r^2}{v^2}\right|^{(1-\mu)/2\mu} v^{-1} dv < \tfrac{1}{4} |r|^{(1-\mu)/2\mu} \int_0^{\infty} |q-1|^{(1-\mu)/2\mu} q^{-(3\mu+1)/4\mu} dq < \text{const.} |r|^{(1-\mu)/2\mu}. \tag{2.7.22}$$

From this it follows that $I_2(t)$ is the Fourier transform of a function in $\mathfrak{L}^{\nu}(r)$ $(1 < \nu \leq 2)$. If $\mu = 1$, the same result holds true by a simpler argument. Hence in either case $I_2(t)$ belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)$.

We are now in a situation analogous to the one discussed in section 2.6.4. The function $\exp(-iH_a t)f_a$ can be written in the form $\sum_i f_i(t)$, as in eq. (2.6.34). The functions $||V_{pq} f_i(t)||$ satisfy eq. (2.6.35). Hence $||V_{pq}\exp(-iH_a t)f_a||$ is integrable over $-\infty < t < -1$ and $1 \le t < \infty$. Also, since f_a belongs to $\mathfrak{D}(H_0)$, $||V_{pq}\exp(-iH_a t)f_a||$ is a bounded function of t. From this it follows as before that $||V_a\exp(-iH_a t)f_a||$ is bounded and belongs to $\mathfrak{L}(t)$. Hence the combination V_a, f_a is admissible, as we wished to show.

Thus far it has been assumed that the sum in eq. (2.7.1) consists of only one term. However, if both V_a, f_{a1} and V_a, f_{a2} are admissible, so is $V_a, f_{a1} + f_{a2}$. Hence in eq. (2.7.1) we may admit any finite number of terms. In other words, V_a, f_a is admissible whenever f_a belongs to $\mathfrak{G}_a$ and each function Q_{pq} satisfies eq. (2.7.5). This result can easily be extended to functions Q_{pq} which satisfy

$$\int_0^R Q_{pq}(x_a)x_a^2 dx_a < \infty, \quad Q_{pq}(x_a) < \text{const.} x_a^{-2-\eta} \quad (x_a > R) \tag{2.7.23}$$

for some positive η. The proof is omitted since, after the foregoing, it is completely straightforward.

2.7.3. Examples of admissible interactions

In the present section we give a short summary of some sufficient conditions on V_{pq} under which eq. (2.7.5) or eq. (2.7.23) is satisfied. It is obvious that, if V_{pq} is square-integrable, eq. (2.7.5) holds true for $\mu = 1$. To obtain a sufficient condition characterized by $\mu = 2$, we observe that

$$[Q_{pq}(x_a)]^2 < 4\pi \int [W_{pq}(\boldsymbol{x}_a)]^2 d\omega_{x_a}. \tag{2.7.24}$$

Hence eq. (2.7.5) holds true for $\mu = 2$ if

$$\int [W_{pq}(\boldsymbol{x}_a)x_a]^2 d\boldsymbol{x}_a < \infty. \tag{2.7.25}$$

To reduce this inequality to a condition on V_{pq}, we require some information as regards the asymptotic behaviour of φ_a. If this is such that eq. (2.6.49) is satisfied for $\alpha = \frac{1}{2}$, eq. (2.7.25) holds true whenever

$$\int [V_{pq}(\boldsymbol{X})]^4 X^2 d^3\boldsymbol{X} < \infty, \qquad \int [V_{pq}(\boldsymbol{X})]^4 d^3\boldsymbol{X} < \infty. \tag{2.7.26}$$

If it is known that in eq. (2.6.49) α may be as large as $\frac{1}{2} + \frac{1}{2}\eta$, with some positive η, the condition V_{pq} has to satisfy at infinity can further be relaxed. In this case it is sufficient if

$$\int_0^R [V_{pq}(\boldsymbol{X})]^2 d^3\boldsymbol{X} < \infty, \quad |V_{pq}(\boldsymbol{X})| < \text{const.} X^{-1-\frac{1}{2}\eta} \quad (X > R). \tag{2.7.27}$$

These statements are easily checked with the methods given in section 2.6.6.

2.7.4. The scattering operators in momentum space

Under the assumptions that f_a belongs to $\mathfrak{G}_a$ and that the interaction V_a is a linear combination of terms satisfying either eq. (2.7.5) or eq. (2.7.23), we proceed to express the quantity $(g_b, S_{ba} f_a)$ in terms of the Fourier transforms $\hat{f}$ and $\hat{g}$. This is most easily done with the help of section 2.6.7. It is obvious that for section 2.6.7 to be valid, it is essential that Ω_{a+} and Ω_{b-} exist. Apart from this we merely need the fact that $\|V_a \exp(-iH_a t) f_a\|$ is a bounded and integrable function of t, and, in connection with eq. (2.6.64), that $V_a P_a(X_a)\varphi_a$ belongs to $\mathfrak{L}^2$. Since these conditions are satisfied under the present assumptions, section 2.6.7 can be copied unchanged. In particular, if eq. (2.6.67) is combined with eq. (2.7.1), we obtain

$$\left.\begin{aligned} (g_b, S_{ba} f_a) &= \delta_{ba}(g_b, f_a) \\ &+ \tfrac{1}{4} i^{l+1} (2\pi)^{-\frac{3}{2}m_b + \frac{5}{2}} \lim_{\varepsilon \to 0} \lim_{X_b \to \infty} \sum_{l=0}^{L} \sum_m \int dE \int d\omega_{k_b} k_b^{3m_b - 5} k_a^{\frac{1}{2}} \bar{\hat{g}}(\boldsymbol{k}_b) F(\boldsymbol{k}_b, l, m; \varepsilon; X_b) \hat{f}_{lm}(k_a), \\ F(\boldsymbol{k}_b, l, m; \varepsilon; X_b) &= (P_b(X_b)\varphi_b e^{i\boldsymbol{k}_b \cdot \boldsymbol{x}_b}, [-1 + V_b R(E + i\varepsilon)] V_a \varphi_a \frac{1}{\sqrt{x_a}} J_{l+\frac{1}{2}}(k_a x_a) Y_{lm}(\omega_{x_a})). \end{aligned}\right\} \tag{2.7.28}$$

In this expression the limit with respect to X_a has been performed in the integrand. This is permitted since under the present assumptions

$$V_a(\boldsymbol{x}_a', \boldsymbol{x}_a)\varphi_a(\boldsymbol{x}_a') \frac{1}{\sqrt{x_a}} J_{l+\frac{1}{2}}(k_a x_a) Y_{lm}(\omega_{x_a}) \tag{2.7.29}$$

belongs to $\mathfrak{L}^2(\boldsymbol{x}_a', \boldsymbol{x}_a)$.

Thus far eq. (2.7.28) is restricted to functions f_a in $\mathfrak{G}_a$. We now show that, if, given V_a, there is an equation of the form (2.7.28) for any particular f_a, then there is a similar equation for $E_a(K^2) f_a$. Since in general $E_a(K^2) f_a$ does not belong to $\mathfrak{G}_a$, the validity of eq. (2.7.28) is thus extended to a larger class of functions f_a. This is of practical importance in future sections.

The proof of our assertion is based on a straightforward generalization of eq. (2.2.28), according to which

$$E_b(K^2) S_{ba} = S_{ba} E_a(K^2). \tag{2.7.30}$$

Let us now assume that V_a and f_a are such that $(g_b, S_{ba} f_a)$ can be evaluated with the help of eq. (2.7.28). Then $(E_b(K^2) g_b, S_{ba} f_a)$ can also be evaluated with eq. (2.7.28). This is due to the fact that we do not have to impose any special restrictions on $\hat{g}$, apart from its belonging to $\mathfrak{L}^2$. Now according to eq. (2.5.21), the operator $E_b(K^2)$ transforms $\hat{g}(\boldsymbol{k}_b)$ into a function which is equal to $\hat{g}(\boldsymbol{k}_b)$ if $0 < k_b < (K^2 - \lambda_b)^{\frac{1}{2}}$ and vanishes if $k_b > (K^2 - \lambda_b)^{\frac{1}{2}}$. Hence, to find $(E_b(K^2)g_b, S_{ba}f_a)$ with the help of eq. (2.7.28), we merely have to restrict the integration with respect to E to the interval $\max(\lambda_a, \lambda_b) < E \leq K^2$ (cf. eq. (2.6.65)). But this restricting the integration is tantamount to replacing f_a by $E_a(K^2)f_a$. Hence we are in effect replacing f_a by $E_a(K^2)f_a$, and we are evaluating $(g_b, S_{ba}E_a(K^2)f_a)$, by eq. (2.7.30). This can thus be done with eq. (2.7.28), as we wished to show.

It is appropriate to call a function defined in $0 \leq k < \infty$ a step-function if it is constant in some interval $0 < k < K$ and vanishes if $k > K$. In this terminology, we may say that, if f_a belongs to $\mathfrak{G}_a$, so that $\hat{f}_{lm}$ belongs to $\hat{\mathfrak{G}}_{lm}$, the operator $E_a(K^2)$ transforms $\hat{f}_{lm}$ into a step-function times a function in $\hat{\mathfrak{G}}_{lm}$. We now denote the set consisting of all linear combinations of step-functions by $\mathfrak{S}$. If $\hat{f}_{lm}$ is the product of a function in $\mathfrak{S}$ times a function in $\hat{\mathfrak{G}}_{lm}$, we say that it belongs to $\mathfrak{S}\hat{\mathfrak{G}}_{lm}$. Likewise, if $\hat{f}$ is of the form (2.7.1) and each $\hat{f}_{lm}$ belongs to $\mathfrak{S}\hat{\mathfrak{G}}_{lm}$, we say that $\hat{f}$ belongs to $\mathfrak{S}\hat{\mathfrak{G}}$, f to $\mathfrak{S}\mathfrak{G}$, and f_a to $\mathfrak{S}\mathfrak{G}_a$. It will be observed that $\mathfrak{S}\mathfrak{G}_a$ contains $\mathfrak{G}_a$ as a subset. It follows from the foregoing that for eq. (2.7.28) to be valid, it is sufficient if f_a belongs to $\mathfrak{S}\mathfrak{G}_a$ and V_a satisfies eq. (2.7.5) or eq. (2.7.23).

2.7.5. Partial waves

Thus far no assumption has been made as regards the behaviour of the interaction under rotations in coordinate space. The remaining part of this investigation is devoted to systems in which each two-body interaction V_{ij} is spherically symmetric. This case permits a number of interesting simplifications owing to the conservation of angular momentum. In discussing these, we make the additional restriction that the system is split into two fragments both in the initial and in the final channel.

If the interaction is spherically symmetric, it can be assumed without loss of generality that the eigenfunctions $\varphi_{(j)}(\boldsymbol{x}_j)$ are also eigenfunctions of angular momentum. For simplicity we even assume in the following that $\varphi_a(\boldsymbol{x}'_a)$ and $\varphi_b(\boldsymbol{x}'_b)$ are eigenfunctions of angular momentum 0. This is a far-reaching restriction, which is, however, not essential. It is made only to separate the subject of the present investigation from problems in the field of Clebsch-Gordan coefficients.

If the eigenfunctions $\varphi_{(j)}(\boldsymbol{x}_j)$ do not correspond to angular momentum 0, a function f_a of the product-form $\varphi_a(\boldsymbol{x}'_a)f(\boldsymbol{x}_a)$ will not be an eigenfunction of the total angular momentum. In this case the eigenvalues $\lambda_{(j)}$ will be degenerate. Hence there will be functions $f_a, f_b, \ldots$ with $\lambda_a = \lambda_b = \ldots$, as in eq. (2.3.5). It may be convenient to use asymptotic wave-functions which are eigenfunctions of angular momentum. This can be achieved if, instead of defining the channels as in eq. (2.3.5), one takes as asymptotic wave-functions suitable linear combinations of functions of the form (2.3.5). This leads to a modification of the formalism which is left to the reader.

Analysing eq. (2.7.28) from the point of view of spherical symmetry, we recall that in channel a we are using a three-dimensional coordinate $\boldsymbol{x}_a$ plus a coordinate $\boldsymbol{x}_a'$ which, in fact, is a set of $n-2$ three-dimensional coordinates $\boldsymbol{x}_j$, where n is the total number of particles. Let us now imagine that there is a three-dimensional coordinate frame in which both $\boldsymbol{x}_j$ ($j = 1,\ldots,n-2$) and $\boldsymbol{x}_a$ are measured, and let us rotate this frame through Euler angles ω. This changes $\boldsymbol{x}_j,\boldsymbol{x}_a$ into $D(\omega)\boldsymbol{x}_j$, $D(\omega)\boldsymbol{x}_a$. Combining $\boldsymbol{x}_j,\boldsymbol{x}_a$ into a $(3n-3)$-dimensional coordinate $\boldsymbol{x}$, it is convenient to write

$$f(D(\omega)\boldsymbol{x}_j,D(\omega)\boldsymbol{x}_a) = f(D(\omega)\boldsymbol{x}) = D(\omega)f(\boldsymbol{x}).$$

In this notation the assumption of spherical symmetry implies

$$D(\omega)V_{ij} = V_{ij}D(\omega), \qquad D(\omega)R(\lambda) = R(\lambda)D(\omega).$$

If φ_a is an eigenfunction of angular momentum with eigenvalue 0,

$$D(\omega)\varphi_a = \varphi_a, \tag{2.7.31}$$

and similarly for φ_b. Also,

$$\left.\begin{aligned} D(\omega)Y_{lm}(\omega_{x_a}) &= \sum_{m'} D^l_{m'm}(\omega)Y_{lm'}(\omega_{x_a}),\\ D(\omega)e^{i\boldsymbol{k}_b\cdot\boldsymbol{x}_b} &= (2\pi)^{\frac{3}{2}}\sum_{l=0}^{\infty}\sum_{m,m'}\frac{i^l}{\sqrt{k_bx_b}}J_{l+\frac{1}{2}}(k_bx_b)\overline{Y}_{lm}(\omega_{k_b})D^l_{m'm}(\omega)Y_{lm'}(\omega_{x_b}),\end{aligned}\right\} \tag{2.7.32}$$

the functions $D^l_{m'm}$ being orthogonal and having norm $2\pi\sqrt{2}(2l+1)^{-\frac{1}{2}}$ (Rose (28) sections 14,16). Now obviously

$$(g,f) = (D(\omega)g,D(\omega)f) = (8\pi^2)^{-1}\int(D(\omega)g,D(\omega)f)d\omega. \tag{2.7.33}$$

Hence, with eq. (2.7.28),

$$\left.\begin{aligned} F(\boldsymbol{k}_b,l,m;\varepsilon;X_b) &= \frac{(-i)^l(2\pi)^{\frac{3}{2}}}{2l+1}\frac{1}{\sqrt{k_b}}Y_{lm}(\omega_{k_b})\\ \times\sum_{m'}\Big(P_b(X_b)\varphi_b\frac{1}{\sqrt{x_b}}J_{l+\frac{1}{2}}(k_bx_b)Y_{lm'}(\omega_{x_b})&,[-1+V_bR(E+i\varepsilon)]V_a\varphi_a\frac{1}{\sqrt{x_a}}J_{l+\frac{1}{2}}(k_ax_a)Y_{lm'}(\omega_{x_a})\Big).\end{aligned}\right\} \tag{2.7.34}$$

A second application of eq. (2.7.33) shows that the inner product in eq. (2.7.34) does not depend on m'. Developing $\hat{g}(\boldsymbol{k}_b)$ in spherical harmonics, we thus obtain

$$\left.\begin{aligned} (g_b,S_{ba}f_a) &= \tfrac{1}{2}\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\sum_{l=0}^{L}\sum_m\int\overline{\hat{g}}_{lm}(k_b)S_{ba}(E,l;\varepsilon;X_b)\hat{f}_{lm}(k_a)\sqrt{k_bk_a}\,dE,\\ S_{ba}(E,l;\varepsilon;X_b) &= \delta_{ba}\\ +\pi i\Big(P_b(X_b)\varphi_b\frac{1}{\sqrt{x_b}}J_{l+\frac{1}{2}}(k_bx_b)Y_{l0}(\omega_{x_b})&,[-1+V_bR(E+i\varepsilon)]V_a\varphi_a\frac{1}{\sqrt{x_a}}J_{l+\frac{1}{2}}(k_ax_a)Y_{l0}(\omega_{x_a})\Big).\end{aligned}\right\} \tag{2.7.35}$$

This equation holds true whenever f_a belongs to $\mathfrak{S}\mathfrak{G}_a$ and V_a satisfies eq. (2.7.5) or eq. (2.7.23). It is obvious that V_b must be such that the operator Ω_{b-} exists. Defining

$$\chi_{bl} = \varphi_b(\boldsymbol{x}_b')\frac{1}{\sqrt{x_b}}J_{l+\frac{1}{2}}(k_b x_b)Y_{l0}(\omega_{x_b}), \tag{2.7.36}$$

we can say that in many cases of practical interest the quantity $V_b\chi_{bl}$ belongs to $\mathfrak{L}^2$. This is particularly so if the sufficient condition (2.4.18) is satisfied. If $V_{bl}\chi_{bl}$ belongs to $\mathfrak{L}^2$, we may write

$$\left.\begin{aligned} (g_b,S_{ba}f_a) &= \tfrac{1}{2}\{\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\}\sum_{l=0}^{L}\sum_m\int\bar{\hat{g}}_{lm}(k_b)\tilde{S}_{ba}(E,l;\varepsilon;X_b)\hat{f}_{lm}(k_a)\sqrt{k_bk_a}\,dE,\\ \tilde{S}_{ba}(E,l;\varepsilon;X_b) &= \delta_{ba}-\pi i(P_b(X_b)\chi_{bl},V_a\chi_{al})+\pi i(V_b\chi_{bl},R(E+i\varepsilon)V_a\chi_{al}),\end{aligned}\right\} \tag{2.7.37}$$

χ_{al} being defined by a relation similar to eq. (2.7.36).

If V_b satisfies eq. (2.7.5) or eq. (2.7.23) and g_b belongs to $\mathfrak{S}\mathfrak{G}_b$, it only requires a slight modification of the argument from eq. (2.6.1) onwards to show that in eq. (2.7.35) the function $S_{ba}(E,l;\varepsilon;X_b)$ may be replaced by

$$S_{ba}(E,l;\varepsilon;X_a) = \delta_{ba}+\pi i(V_b\chi_{bl},[-1+R(E+i\varepsilon)V_a]P_a(X_a)\chi_{al}). \tag{2.7.38}$$

The limit with respect to X_b has now already been performed in the integrand, but there appears a limit with respect to X_a. The fact that this replacement yields the same limit as before is directly connected with eq. (2.6.8). If both V_a and V_b satisfy eq. (2.7.5) or eq. (2.7.23) and f_a and g_b belong to $\mathfrak{S}\mathfrak{G}_a$ and $\mathfrak{S}\mathfrak{G}_b$, respectively, we have

$$\left.\begin{aligned} (g_b,S_{ba}f_a) &= \tfrac{1}{2}\{\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\lim_{X_a\to\infty}\}\sum_{l=0}^{L}\sum_m\int\bar{\hat{g}}_{lm}(k_b)\tilde{\tilde{S}}_{ba}(E,l;\varepsilon;X_b,X_a)\hat{f}_{lm}(k_a)\sqrt{k_bk_a}\,dE,\\ \tilde{\tilde{S}}_{ba}(E,l;\varepsilon;X_b,X_a) &= \delta_{ba}-\tfrac{1}{2}\pi i(P_b(X_b)\chi_{bl},V_a\chi_{al})-\tfrac{1}{2}\pi i(V_b\chi_{bl},P_a(X_a)\chi_{al})\\ &\quad+\pi i(V_b\chi_{bl},R(E+i\varepsilon)V_a\chi_{al}).\end{aligned}\right\} \tag{2.7.39}$$

2.7.6. The scattering matrix

Let us consider eq. (2.7.35) for the special case

$$\left.\begin{aligned} \hat{f}_{lm}(k_a) &= 1, & \hat{g}_{lm}(k_b) &= \sqrt{k_a/k_b} & &(\max(\lambda_a,\lambda_b)\le E_1<E<E_2),\\ \hat{f}_{lm}(k_a) &= 0, & \hat{g}_{lm}(k_b) &= 0 & &(E<E_1,\quad E>E_2).\end{aligned}\right\} \tag{2.7.40}$$

This is compatible with the condition that $\hat{f}_{lm}$ must belong to $\mathfrak{S}\hat{\mathfrak{G}}_{lm}$. In view of the inequality

$$|(g_b,S_{ba}f_a)|\le\|g_b\|\,\|f_a\| \tag{2.7.41}$$

it yields

$$|\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\int_{E_1}^{E_2} S_{ba}(E,l;\varepsilon;X_b)k_a dE| \le \int_{E_1}^{E_2} k_a dE. \tag{2.7.42}$$

If we now define

$$\Sigma_{ba}(E,l) = \lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\int_{\max(\lambda_a,\lambda_b)}^{E} S_{ba}(E',l;\varepsilon;X_b)k'_a dE', \tag{2.7.43}$$

eq. (2.7.42) shows that $\Sigma_{ba}(E,l)$ is an absolutely continuous function of E. Hence there exists a function $S_{ba}(E,l)k_a$ such that

$$\Sigma_{ba}(E,l) = \int_{\max(\lambda_a,\lambda_b)}^{E} S_{ba}(E',l)k'_a dE', \tag{2.7.44}$$

$S_{ba}(E,l)k_a$ being the derivative of $\Sigma_{ba}(E,l)$ for almost every E in the interval

$$\max(\lambda_a,\lambda_b) \le E < \infty. \tag{2.7.45}$$

According to eq. (2.7.42)

$$\left|\int_{E_1}^{E_2} S_{ba}(E,l)k_a dE\right| \le \int_{E_1}^{E_2} k_a dE. \tag{2.7.46}$$

It is shown in section 2.7.8 that from this it follows that

$$|\mathrm{Re}S_{ba}(E,l)| \le 1, \qquad |\mathrm{Im}S_{ba}(E,l)| \le 1 \tag{2.7.47}$$

almost everywhere in the interval (2.7.45).

Let us now consider a function $\hat{f}_{lm}(k_a)$ which takes the value 1 in $E_1 < E < E_2$ and is sufficiently smooth to belong to $\mathfrak{G}_{lm}$. If f_a stands for φ_a times the Fourier transform of $\hat{f}_{lm}(k_a)Y_{lm}(\omega_{k_a})$ and V_a satisfies eq. (2.7.5) or (2.7.23), then $||V_a\exp(-iH_a t)f_a||$ is bounded and belongs to $\mathfrak{L}(t)$. If g_b is equal to φ_b times the Fourier transform of $\hat{g}_{lm}(k_b)Y_{lm}(\omega_{k_b})$, where $\hat{g}_{lm}$ is the same function as in eq. (2.7.40), we have

$$\left.\begin{aligned}\tfrac{1}{2}\int_{E_1}^{E_2} S_{ba}(E,l;\varepsilon;X_b)k_a dE &= \delta_{ba}(g_b,f_a)\\ -\int_{-\infty}^{\infty}\Big[i(g_b,e^{iH_b t}P_b(X_b)V_a e^{-iH_a t}f_a) &+ \int_0^{\infty} e^{-\varepsilon s}(g_b,e^{iH_b(s+t)}P_b(X_b)V_b e^{-iHs}V_a e^{-iH_a t}f_a)ds\Big]dt.\end{aligned}\right\} \tag{2.7.48}$$

This follows from eq. (2.6.1) and the beginning of section 2.6.7.

As long as ε is positive, either side of eq. (2.7.48) is bounded uniformly in X_b, by the argument of eq. (2.6.60). If E_2 is held fixed, $||H_0 g_b||$ does not exceed $||g_b||$

times a constant determined by E_2. Hence, in view of eq. (2.6.60), either side of eq. (2.7.48) is less than

$$\text{const.}||g_b||/\varepsilon = \text{const.}\left[\int_{E_1}^{E_2} k_a dE\right]^{\frac{1}{2}}/\varepsilon. \tag{2.7.49}$$

If X_b tends to ∞, the second term in the square brackets in eq. (2.7.48) tends to

$$i(g_b, e^{iH_bt}[\Omega^*_{b-,\varepsilon} - 1]V_a e^{-iH_at} f_a). \tag{2.7.50}$$

Hence, since $||\Omega_{b-,\varepsilon}||$ is bounded uniformly in ε,

$$\left|\lim_{X_b\to\infty}\int_{E_1}^{E_2} S_{ba}(E,l;\varepsilon;X_b)k_a dE\right| < \text{const.}\left[\int_{E_1}^{E_2} k_a dE\right]^{\frac{1}{2}}. \tag{2.7.51}$$

We want to use these estimates to study the relation

$$\left.\begin{aligned}\int_{E_1}^{E_2} S_{ba}(E,l;\varepsilon;X_b)\sqrt{k_bk_a}\,dE &= \frac{\sqrt{k_b}}{\sqrt{k_a}}\bigg|_{E_2}\int_{E_1}^{E_2} S_{ba}(E,l;\varepsilon;X_b)k_a dE \\ &+ \tfrac{1}{4}\int_{E_1}^{E_2}\frac{\sqrt{k_b}}{\sqrt{k_a}}\left(\frac{1}{k_b^2} - \frac{1}{k_a^2}\right)\left[\int_{E_1}^{E} S_{ba}(E',l;\varepsilon;X_b)k'_a dE'\right]dE.\end{aligned}\right\} \tag{2.7.52}$$

If $k_a > 0$ at the point $E = E_1$, eq. (2.7.52) is straightforward. If $E_1 = \lambda_a$, there might be convergence difficulties. However, let ε be positive. The estimate (2.7.49) then shows that, even if $E_1 = \lambda_a$, eq. (2.7.52) is completely correct. In the second term on the right the integrand is less than an integrable function of E which does not depend on X_b. Hence in this particular term

$$\lim_{X_b\to\infty}\int_{E_1}^{E_2}\left[\int_{E_1}^{E} dE'\right]dE = \int_{E_1}^{E_2}\lim_{X_b\to\infty}\left[\int_{E_1}^{E} dE'\right]dE, \tag{2.7.53}$$

by the theorem of dominated convergence. With the help of eq. (2.7.51) the argument can now be extended to show that in fact

$$\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\int_{E_1}^{E_2}\left[\int_{E_1}^{E} dE'\right]dE = \int_{E_1}^{E_2}\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\left[\int_{E_1}^{E} dE'\right]dE. \tag{2.7.54}$$

Combing this result with eq. (2.7.52) yields

$$\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\int_{E_1}^{E_2} S_{ba}(E,l;\varepsilon;X_b)\sqrt{k_bk_a}\,dE = \int_{E_1}^{E_2} S_{ba}(E,l)\sqrt{k_bk_a}\,dE. \tag{2.7.55}$$

Now let $\hat{f}_{lmM}$ and $\hat{g}_{lmN}$ be any two functions in $\mathfrak{S}\hat{\mathfrak{G}}_{lm}$. Integration by parts as in eq. (2.7.52) shows that

$$\left.\begin{aligned}\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\int \bar{\hat{g}}_{lmN}(k_b)S_{ba}(E,l;\varepsilon;X_b)\hat{f}_{lmM}(k_a)\sqrt{k_bk_a}\,dE\\ = \int \bar{\hat{g}}_{lmN}(k_b)S_{ba}(E,l)\hat{f}_{lmM}(k_a)\sqrt{k_bk_a}\,dE.\end{aligned}\right\}\quad(2.7.56)$$

Hence, in view of eq. (2.7.35),

$$(g_{bN},S_{ba}f_{aM}) = \tfrac{1}{2}\int \bar{\hat{g}}_{lmN}(k_b)S_{ba}(E,l)\hat{f}_{lmM}(k_a)\sqrt{k_bk_a}\,dE. \quad (2.7.57)$$

If $\hat{f}_{lm}$ is any function satisfying eq. (2.7.2), there is a sequence $\hat{f}_{lmM}$ in $\mathfrak{S}\hat{\mathfrak{G}}_{lm}$ such that

$$\lim_{M\to\infty}\int_0^\infty |\hat{f}_{lm}(k_a) - \hat{f}_{lmM}(k_a)|^2 k_a^2 dk_a = 0, \quad (2.7.58)$$

and similarly for $\hat{g}_{lm}$. Now S_{ba} is a bounded operator. Also, $S_{ba}(E,l)$ is a bounded function, by eq. (2.7.47). Letting M and N tend to ∞, we thus obtain

$$\lim_{N,M\to\infty}(g_{bN},S_{ba}f_{aM}) = (g_b,S_{ba}f_a) = \tfrac{1}{2}\int \bar{\hat{g}}_{lm}(k_b)S_{ba}(E,l)\hat{f}_{lm}(k_a)\sqrt{k_bk_a}\,dE, \quad (2.7.59)$$

even for the most general functions $\hat{f}_{lm}$ and $\hat{g}_{lm}$ we may want to consider.

With eq. (2.7.35) it follows that

$$\left.\begin{aligned}\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\int \bar{\hat{g}}_{lm}(k_b)S_{ba}(E,l;\varepsilon;X_b)\hat{f}_{lm}(k_a)\sqrt{k_bk_a}\,dE\\ = \int \bar{\hat{g}}_{lm}(k_b)S_{ba}(E,l)\hat{f}_{lm}(k_a)\sqrt{k_bk_a}\,dE\end{aligned}\right\}\quad(2.7.60)$$

whenever $\hat{f}_{lm}$ belongs to $\mathfrak{S}\hat{\mathfrak{G}}_{lm}$ and $\hat{g}_{lm}$ satisfies a relation of the form (2.7.2). Here it is understood that V_a satisfies eq. (2.7.5) or eq. (2.7.23). If, in addition, V_b is such that $V_b\chi_{bl}$ belongs to $\mathfrak{L}^2$, the function $S_{ba}(E,l;\varepsilon;X_b)$ may be replaced by the function $\tilde{S}_{ba}(E,l;\varepsilon;X_b)$ defined in eq. (2.7.37).

If in eq. (2.7.59) we choose in particular

$$\left.\begin{aligned}\hat{g}_{lm}(k_b) &= \hat{f}_{lm}(k_a) = 0 \qquad (E < \max(\lambda_a,\lambda_b)),\\ \hat{g}_{lm}(k_b) &= S_{ba}(E,l)\hat{f}_{lm}(k_a)\sqrt{k_a/k_b} \qquad (E > \max(\lambda_a,\lambda_b)),\end{aligned}\right\}\quad(2.7.61)$$

the inequality (2.7.41) gives

$$\int_{\max(\lambda_a,\lambda_b)} |S_{ba}(E,l)\hat{f}_{lm}(k_a)|^2 k_a dE \le \int_{\max(\lambda_a,\lambda_b)} |\hat{f}_{lm}(k_a)|^2 k_a dE. \quad (2.7.62)$$

Since this must hold true whenever the right-hand side is finite, it follows that

$$|S_{ba}(E,l)| \leq 1 \tag{2.7.63}$$

almost everywhere in the interval (2.7.45).

2.7.7. Properties of the scattering matrix

In the present section we assume that V_b satisfies eq. (2.4.25), so that all functions in $\mathfrak{C}_b$ are of the form $\varphi_b(\boldsymbol{x}_b')h(\boldsymbol{x}_b)$. Then

$$S_{ba}\varphi_a(\boldsymbol{x}_a')\int e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a}\hat{f}_{lm}(k_a)Y_{lm}(\omega_{k_a})d\boldsymbol{k}_a = \varphi_b(\boldsymbol{x}_b')\int e^{i\boldsymbol{k}_b\cdot\boldsymbol{x}_b}\hat{h}_{lm}(k_b)Y_{lm}(\omega_{k_b})d\boldsymbol{k}_b, \tag{2.7.64}$$

where $\hat{h}_{lm}$ satisfies

$$\int\bar{\hat{g}}_{lm}(k_b)\hat{h}_{lm}(k_b)k_b dE = \int\bar{\hat{g}}_{lm}(k_b)S_{ba}(E,l)\hat{f}_{lm}(k_a)\sqrt{k_bk_a}\,dE, \tag{2.7.65}$$

by eq. (2.7.59). Hence

$$\left.\begin{aligned} S_{ba}\varphi_a(\boldsymbol{x}_a')\int e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a}\hat{f}_{lm}(k_a)Y_{lm}(\omega_{k_a})d\boldsymbol{k}_a \\ = \varphi_b(\boldsymbol{x}_b')\int e^{i\boldsymbol{k}_b\cdot\boldsymbol{x}_b}S_{ba}(E,l)\hat{f}_{lm}(k_a)Y_{lm}(\omega_{k_b})\sqrt{k_a/k_b}\,d\boldsymbol{k}_b. \end{aligned}\right\} \tag{2.7.66}$$

If $\lambda_a < \lambda_b$ and $\hat{f}_{lm}(k_a)$ vanishes except in the interval $\lambda_a \leq E \leq \lambda_b$, the right-hand side of eq. (2.7.65) vanishes. This means that $S_{ba}f_a = 0$. It is obvious from eq. (2.6.67) that this situation arises whenever $\hat{f}(k_a)$ vanishes outside $\lambda_a \leq E \leq \lambda_b$. It does not matter whether in channels a and b the system is split into only two fragments. It is therefore appropriate to call λ_b the threshold for scattering into channel b, channel b being open or closed according as $E > \lambda_b$ or $E < \lambda_b$.

Now let $\hat{f}_{lm}(k_a)$ be zero except in an interval I in which there are no thresholds. Let the channels which are open in I all refer to splittings into two fragments, the respective functions φ_b all having angular momentum 0. The scattering with initial state f_a can then be described completely in terms of a set of scattering functions $S_{ba}(E,l)$, the parameter b running through all open channels. Let us now assume that there is unitarity in the sense of section 2.3.5. If eq. (2.3.33) holds true we have

$$\sum_b(S_{bc}g_c,S_{ba}f_a) = \delta_{ca}(g_c,f_a). \tag{2.7.67}$$

With eq. (2.7.66) this yields

$$\sum_b\int_I\bar{\hat{g}}_{lm}(k_c)\bar{S}_{bc}(E,l)S_{ba}(E,l)\hat{f}_{lm}(k_a)\sqrt{k_ck_a}\,dE = \delta_{ca}\int_I\bar{\hat{g}}_{lm}(k_c)\hat{f}_{lm}(k_a)\sqrt{k_ck_a}\,dE. \tag{2.7.68}$$

Hence, since $\hat{f}_{lm}$ and $\hat{g}_{lm}$ are arbitrary,

$$\sum_b \bar{S}_{bc}(E,l)S_{ba}(E,l) = \delta_{ca}, \qquad (2.7.69)$$

it being understood that E is restricted to a certain interval I in which the channels a and c are open, the summation including all open channels and no closed ones.

In the present problem it is useful to consider the conjugation C which transforms f_a into $\bar{f}_a$. This can be discussed along the lines of section 2.3.6. Since φ_a refers to angular momentum 0, it may be assumed without loss of generality that φ_a is real. Hence in eq. (2.3.54) $a' = a$. Therefore,

$$(g_b, \Omega^*_{b+}\Omega_{a-}f_a) = (C\Omega_{a-}f_a, C\Omega_{b+}g_b) = (\Omega_{a+}\bar{f}_a, \Omega_{b-}\bar{g}_b) = (S_{ba}\bar{f}_a, \bar{g}_b). \quad (2.7.70)$$

Equation (2.7.59) now gives

$$(g_b, \Omega^*_{b+}\Omega_{a-}f_a) = \tfrac{1}{2}\int \bar{g}_{lm}(k_b)\bar{S}_{ba}(E,l)\hat{f}_{lm}(k_a)\sqrt{k_b k_a}\, dE. \qquad (2.7.71)$$

This shows that, if $S_{ba}(E,l)$ corresponds to $\Omega^*_{b-}\Omega_{a+}$, the function $\bar{S}_{ba}(E,l)$ corresponds to $\Omega^*_{b+}\Omega_{a-}$. Also, since the left-hand side of eq. (2.7.71) is nothing but $(S_{ab}g_b, f_a)$,

$$S_{ab}(E,l) = S_{ba}(E,l) \qquad (2.7.72)$$

almost everywhere in the interval (2.7.45). Combining this result with eq. (2.7.69), we see that, if in the interval I the functions $S_{ba}(E,l)$ are considered as the elements of a matrix $\mathcal{S}(E,l)$, this matrix is unitary and symmetric.

The symmetry of $\mathcal{S}(E,l)$ is simply due to the Hamiltonian commuting with the conjugation C. It must be stressed that, to obtain the unitarity, we had to assume that eq. (2.3.33) holds true. Now it was remarked already at the end of section 2.3.4 that we do not know what conditions on the interaction are sufficient for eq. (2.3.33) to be satisfied. Hence we do not really have any insight into the question of unitarity.

2.7.8. An auxiliary formula

We must still justify eq. (2.7.47). This has been used to obtain eq. (2.7.59) and is thus an essential step in the argument. Simplifying the notation of eq. (2.7.46), we consider a function $S(E)$ which is defined in an interval I_0 and satisfies

$$\left|\int_I S(E)dE\right| < m(I), \qquad (2.7.73)$$

I denoting any particular interval contained in I_0, and $m(I)$ its measure. Decomposing S into its real and imaginary parts, $S = A + iB$, we want to show that $|A| \le 1$, $|B| \le 1$ almost everywhere in I_0.

Let us now suppose that $|A| > 1$ in a set U of positive measure. Let us suppose in particular that $A > 1$ in U. Then

$$\int_U A(E)dE > m(U) > 0. \qquad (2.7.74)$$

From this it follows that there is a positive number η such that

$$\int_U A(E)dE > m(U) + 2\eta. \tag{2.7.75}$$

The set U is not necessarily an interval, hence there is not yet a contradiction with eq. (2.7.73). However, we can choose a sequence O_n of open sets all containing U such that $m(O_n)$ tends to $m(U)$ as n tends to ∞. If n exceeds some N, the set O_n satisfies

$$m(O_n) < m(U) + \eta. \tag{2.7.76}$$

An open set being the sum of a denumerable set of open intervals, it follows from eq. (2.7.73) that, if $n > N$,

$$\left|\int_{O_n} A(E)dE\right| < m(O_n) < m(U) + \eta. \tag{2.7.77}$$

Also, since A is integrable,

$$\left|\int_{O_n - U} A(E)dE\right| < \eta, \tag{2.7.78}$$

provided n exceeds some M. Hence, if $n > M$,

$$\left|\int_{O_n} A(E)dE\right| \geq \int_U A(E)dE - \left|\int_{O_n - U} A(E)dE\right| > m(U) + \eta. \tag{2.7.79}$$

Since this is incompatible with eq. (2.7.77), it follows that A cannot exceed 1 in a set of positive measure. By a similar argument, it cannot be less than -1 in a set of positive measure. Hence $|A| \leq 1$ almost everywhere in I_0, and similarly for B. This completes the proof of eq. (2.7.47).

2.8. The scattering of a beam

2.8.1. Sums of partial waves

We conclude the present investigation with a discussion of the function

$$\left.\begin{aligned} & F_{ba}(\boldsymbol{k}_b, \boldsymbol{k}_a; \varepsilon; X_b) \\ & = -(P_b(X_b)\varphi_b e^{i\boldsymbol{k}_b\cdot\boldsymbol{x}_b}, V_a\varphi_a e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a}) + (V_b\varphi_b e^{i\boldsymbol{k}_b\cdot\boldsymbol{x}_b}, R(E+i\varepsilon)V_a\varphi_a e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a}), \end{aligned}\right\} \tag{2.8.1}$$

it being assumed that both $V_a\varphi_a$ and $V_b\varphi_b$ belong to $\mathfrak{L}^2$. It follows from eq. (2.6.67) that the function F_{ba} is an intermediate step in evaluating $(g_b, S_{ba}f_a)$. It also occurs in the expression for $||(S_{ba} - \delta_{ba})f_a||$, according to eq. (2.6.71). In view of this, we want to study the limiting behaviour of certain integrals which have F_{ba} in their integrands. This is done first from a formal point of view. In section 2.8.4 our results on the mathematical properties of F_{ba} will make it possible to define the scattering amplitude. The physical interpretation of this quantity is discussed in sections 2.8.6 to 2.8.8, where it is shown to describe the scattering through fixed angles of beams of projectiles.

In the following it is assumed throughout that in channels a and b the system is split into two fragments. Each interaction V_{ij} is spherically symmetric, both φ_a and φ_b are eigenfunctions of angular momentum 0. The functions $V_a\varphi_a$ and $V_b\varphi_b$ belong to $\mathfrak{L}^2$, a further restriction being imposed in eq. (2.8.18).

Under the present assumptions we have

$$V_a\varphi_a e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a} = \underset{N\to\infty}{\mathrm{l.i.m.}}\,(2\pi)^{\frac{3}{2}} V_a\varphi_a \sum_{l=0}^{N}\sum_m \frac{i^l}{\sqrt{k_a x_a}} J_{l+\frac{1}{2}}(k_a x_a)\bar{Y}_{lm}(\omega_{k_a})Y_{lm}(\omega_{x_a}), \tag{2.8.2}$$

and similarly for channel b. If we write

$$\cos\vartheta = (\boldsymbol{k}_b\cdot\boldsymbol{k}_a)/k_b k_a, \tag{2.8.3}$$

it follows with the methods of section 2.7.5 that

$$\left.\begin{array}{c} F_{ba}(\boldsymbol{k}_b,\boldsymbol{k}_a;\varepsilon;X_b) = \\ \dfrac{2\pi^2}{\sqrt{k_b k_a}}\displaystyle\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)[-(P_b(X_b)\chi_{bl},V_a\chi_{al}) + (V_b\chi_{bl},R(E+i\varepsilon)V_a\chi_{al})]. \end{array}\right\} \tag{2.8.4}$$

In obtaining this result, use was made of the relation

$$\sum_m Y_{lm}(\omega_{k_b})\bar{Y}_{lm}(\omega_{k_a}) = \frac{2l+1}{4\pi}P_l(\cos\vartheta). \tag{2.8.5}$$

For future reference we note that the sum in eq. (2.8.4) converges absolutely.

It is obvious from eq. (2.8.4) that, apart from ε and X_b, F_{ba} depends only on the angle ϑ between $\boldsymbol{k}_a$ and $\boldsymbol{k}_b$, and on the variable E, which is related to k_a and k_b according to eq. (2.6.65). We therefore define

$$F_{ba}(E,\vartheta;\varepsilon;X_b) = F_{ba}(\boldsymbol{k}_b,\boldsymbol{k}_a;\varepsilon;X_b). \tag{2.8.6}$$

In view of eq. (2.7.37) we may write

$$F_{ba}(E,\vartheta;\varepsilon;X_b) = \frac{2\pi i}{\sqrt{k_b k_a}}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)[\delta_{ba} - \hat{S}_{ba}(E,l;\varepsilon;X_b)], \tag{2.8.7}$$

this series being absolutely convergent.

Now let $\hat{f}(k_a)$ and $\hat{g}(k_b)$ be any two functions in $\hat{\mathfrak{G}}_{lm}$. Let us define

$$\left.\begin{aligned} \hat{f}_l(\boldsymbol{k}_a) &= \hat{f}(k_a)Y_{l0}(\omega_{k_a}),\\ f_l(\boldsymbol{x}_a) &= (2\pi)^{-\frac{3}{2}}\int e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a}\hat{f}_l(\boldsymbol{k}_a)d\boldsymbol{k}_a,\\ f_{al}(\boldsymbol{x}'_a,\boldsymbol{x}_a) &= \varphi_a(\boldsymbol{x}'_a)f_l(\boldsymbol{x}_a), \end{aligned}\right\} \quad (2.8.8)$$

and let us define $g_{bl}(\boldsymbol{x}'_b,\boldsymbol{x}_b)$ in a similar way. Then it is clear that f_{al} belongs to $\mathfrak{G}_a$. Hence, since V_a is square-integrable by assumption, the combination V_a,f_{al} is admissible in the sense of section 2.7.1. Since V_b is also square-integrable, the quantity $(g_{bl},S_{ba}f_{al})$ can be evaluated with the help of eq. (2.7.37).

Under suitable restrictions on V_a and V_b, we first study the integral

$$J(\varepsilon) = \int\bar{\hat{g}}(k_b)(V_b\varphi_b e^{i\boldsymbol{k}_b\cdot\boldsymbol{x}_b}, R(E+i\varepsilon)V_a\varphi_a e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a})\hat{f}(k_a)k_bk_a dE. \quad (2.8.9)$$

This can be decomposed into angular-momentum components with the help of eq. (2.8.4). From eq. (2.8.7) it is clear that the component l is fairly closely related to $(g_{bl},S_{ba}f_{al})$. If the way is remembered in which the expression (2.7.37) for $(g_{bl},S_{ba}f_{al})$ was derived from eq. (2.6.1) plus a similar equation for $(g_{bl},\Omega^*_{b-}\Omega_{a-}f_{al})$, it follows that

$$\left.\begin{aligned} J(\varepsilon) &= 2\pi^2\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)\int\bar{\hat{g}}(k_b)(V_b\chi_{bl},R(E+i\varepsilon)V_a\chi_{al})\hat{f}(k_a)\sqrt{k_bk_a}\,dE\\ &= 4\pi i\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)\int_{-\infty}^{\infty}dt\int_0^{\infty}ds e^{-\varepsilon s}(g_{bl},e^{iH_b(s+t)}V_b e^{-iHs}V_a e^{-iH_a t}f_{al}). \end{aligned}\right\} \quad (2.8.10)$$

Hence

$$|J(\varepsilon)| \leq 4\pi\sum_{l=0}^{\infty}(2l+1)\int_{-\infty}^{\infty}||V_b e^{-iH_b s}g_{bl}||ds\int_{-\infty}^{\infty}||V_a e^{-iH_a t}f_{al}||dt, \quad (2.8.11)$$

assuming the series on the right to be convergent.

2.8.2. A convergence problem

The function $J(\varepsilon)$ is the sum of a series each term of which is known to have a limit as ε tends to 0. If we can show that the series converges uniformly with respect to ε, it follows that we have

$$\lim_{\varepsilon\to 0}J(\varepsilon) = \lim_{\varepsilon\to 0}\sum_{l=0}^{\infty} = \sum_{l=0}^{\infty}\lim_{\varepsilon\to 0}. \quad (2.8.12)$$

A sufficient condition for this relation to be valid is the convergence of the series on the right-hand side of eq. (2.8.11). We therefore proceed to investigate this.

According to eq. (2.7.7)

$$\left.\begin{aligned}&\sum_{l=0}^{\infty}(2l+1)\|V_{pq}e^{-iH_a t}f_{al}\|^2\\ &= \text{const.}\sum_{l=0}^{\infty}(2l+1)\int Q_{pq}(x)x^2dx\left|\int \exp(-ik^2t)\frac{1}{\sqrt{kx}}J_{l+\frac{1}{2}}(kx)\hat{f}(k)k^2dk\right|^2.\end{aligned}\right\}\quad (2.8.13)$$

In this expression the sum with respect to l can be performed explicitly with the help of the addition theorem

$$\pi\sum_{l=0}^{\infty}(l+\tfrac{1}{2})\frac{1}{\sqrt{kk'}}J_{l+\frac{1}{2}}(kx)J_{l+\frac{1}{2}}(k'x) = \frac{\sin|k-k'|x}{|k-k'|} \quad (2.8.14)$$

(Watson (29) section 11.41 eq. (9)). Indeed, since V_{pq} is square-integrable, the function $Q_{pq}(x)x^2$ is integrable. On the right-hand side of eq. (2.8.13) the summation and the integrations may therefore be interchanged. This yields

$$\left.\begin{aligned}&\sum_{l=0}^{\infty}(2l+1)\|V_{pq}e^{-iH_a t}f_{al}\|^2\\ &= \text{const.}\int Q_{pq}(x)x^2dx\int\int \exp[-i(k^2-k'^2)t]\frac{\sin|k-k'|x}{|k-k'|x}\hat{f}(k)\bar{\hat{f}}(k')k^2k'^2dkdk'.\end{aligned}\right\}\quad (2.8.15)$$

Taking into account that $\hat{f}$ vanishes outside a bounded interval, it is now obvious that

$$\sum_{l=0}^{\infty}(2l+1)\|V_{pq}e^{-iH_a t}f_{al}\|^2 < \text{const.} \quad (2.8.16)$$

uniformly in t. A similar argument applies to channel b. Since V_a is nothing but a sum of two-body interactions V_{pq}, it follows that

$$\sum_{l=0}^{\infty}(2l+1)\left[\int_{-1}^{1}\|V_a e^{-iH_a t}f_{al}\|dt\right]^2 < \infty, \quad (2.8.17)$$

and similarly for channel b.

To extend this result so as to prove that the right-hand side of eq. (2.8.11) is finite, we now assume that the two-body interactions V_{pq} that constitute V_a and V_b are such that there is a positive ζ with

$$\int Q_{pq}(x)(1+x)^{1+\zeta}x^2dx < \infty, \quad (2.8.18)$$

a condition which is fulfilled whenever eq. (2.6.49) holds true for $\alpha = (1+\zeta)/2$ and V_{pq} satisfies

$$\int [V_{pq}(\boldsymbol{X})]^2(1+X)^{1+\zeta}d^3\boldsymbol{X} < \infty. \quad (2.8.19)$$

With eq. (2.7.7) we write

$$\left.\begin{aligned}&\exp[i\Delta(\boldsymbol{x})t]f_l(\boldsymbol{x})\\ = \frac{i^{l-1}}{4t}Y_{l0}(\omega_x)\int\exp(-ik^2t)\Bigg[&\frac{1}{\sqrt{kx}}J_{l+\frac{1}{2}}(kx)+\sqrt{kx}J_{l-\frac{1}{2}}(kx)-\sqrt{kx}J_{l+\frac{3}{2}}(kx)\\ &+\frac{2k}{\sqrt{kx}}J_{l+\frac{1}{2}}(kx)\frac{d}{dk}\Bigg]\hat{f}(k)dk,\end{aligned}\right\}\quad(2.8.20)$$

hence

$$e^{-iH_at}f_{al} = \sum_{i=1}^{4} f_{li}(t), \quad (2.8.21)$$

say. In connection with the term $f_{l1}(t)$ we first consider

$$\left.\begin{aligned}I_1(t) &\equiv \sum_{l=0}^{\infty}(2l+1)\int Q_{pq}(x)x^2dx\left|\int\exp(-ik^2t)\frac{1}{\sqrt{kx}}J_{l+\frac{1}{2}}(kx)\hat{f}(k)dk\right|^2\\ &= \text{const.}\int Q_{pq}(x)x^2dx\int\int\exp[-i(k^2-k'^2)t]\frac{\sin|k-k'|x}{|k-k'|x}\hat{f}(k)\bar{\hat{f}}(k')dkdk'.\end{aligned}\right\}\quad(2.8.22)$$

Owing to eq. (2.8.18)

$$\int Q_{pq}(x)x^2\frac{\sin|k-k'|x}{|k-k'|x}dx < \text{const.} \quad (2.8.23)$$

uniformly in $k-k'$. Hence, if we go over to the variables

$$k^2-k'^2 = r, \qquad k-k' = w, \quad (2.8.24)$$

we find

$$I_1(t) = \left[\int_{-K^2}^{0}dr\int_{-K}^{\frac{r}{2K}}dw+\int_{0}^{K^2}dr\int_{\frac{r}{2K}}^{K}dw\right]e^{-irt}w^{-1}G(w,r), \quad (2.8.25)$$

with some bounded function G. From this it follows as before that $I_1(t)$ is the Fourier transform of a function in $\mathfrak{L}^{\nu}(r)$ $(1<\nu\leq 2)$, hence that $I_1(t)$ belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)$.

We now discuss the function

$$\left.\begin{aligned}I_2(t) &\equiv \sum_{l=0}^{\infty}(2l+1)\int Q_{pq}(x)x^2dx\left|\int\exp(-ik^2t)\sqrt{kx}J_{l-\frac{1}{2}}(kx)\hat{f}(k)dk\right|^2\\ &= \text{const.}\int Q_{pq}(x)x^2dx\left|\int\exp(-ik^2t)\cos\sqrt{kx}\hat{f}(k)dk\right|^2\\ &+\text{const.}\int Q_{pq}(x)x^3dx\int\int\exp[-i(k^2-k'^2)t]\frac{\sin|k-k'|x}{|k-k'|}\hat{f}(k)\bar{\hat{f}}(k')kk'dkdk'.\end{aligned}\right\}\quad(2.8.26)$$

After the foregoing it is obvious that the first term on the right belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)$. Let us denote the second term by $I_{22}(t)$. If in eq. (2.8.18) we choose ζ in the interval $0 < \zeta < 1$, the inequality

$$|\sin x| \leq x^{\zeta}/\zeta \qquad (x \geq 0,\ 0 < \zeta < 1) \tag{2.8.27}$$

yields

$$I_{22}(t) = \iint \exp[-i(k^2 - k'^2)t] \frac{Z(|k-k'|)}{|k-k'|} f(k)\bar{f}(k')kk'dkdk', \tag{2.8.28}$$

the function Z satisfying

$$|Z(|k-k'|)| < \text{const.}|k-k'|^{\zeta}. \tag{2.8.29}$$

In terms of the variables w and r we thus obtain

$$I_{22}(t) = \left[\int_{-K^2}^{0} dr \int_{-K}^{\frac{r}{2K}} dw + \int_{0}^{K^2} dr \int_{\frac{r}{2K}}^{K} dw\right] e^{-irt} |w|^{-2+\zeta} H(w,r), \tag{2.8.30}$$

H being a bounded function. This shows that $I_{22}(t)$ is the Fourier transform of a function which belongs to the classes $\mathfrak{L}^{\nu}(r)$ with $\nu < (1-\zeta)^{-1}$. It follows that there is a ν in the interval $0 < \nu < 1$ such that $I_{22}(t)$ belongs to $\mathfrak{L}^{\nu/(\nu-1)}(t)$.

The terms $f_3(t)$ and $f_4(t)$ in eq. (2.8.21) can be discussed along exactly the same lines. The general result is therefore that there is a class $\mathfrak{L}^{\nu/(\nu-1)}(t)$ which contains each of the four functions $I_i(t)$. Also, $[I_i(t)]^{\frac{1}{2}}/t$ is integrable over $-\infty < t \leq -1$ and $1 \leq t < \infty$.

From the inequalities of Schwarz and Minkowski it now follows that

$$\left. \begin{aligned} &\left\{\sum_{l=0}^{\infty}(2l+1)\left[\int_1^{\infty} ||V_{pq}e^{-iH_a t} f_{al}||dt\right]^2\right\}^{\frac{1}{2}} \leq \int_1^{\infty}\left[\sum_{l=0}^{\infty}(2l+1)||V_{pq}e^{-iH_a t}f_{al}||^2\right]^{\frac{1}{2}} dt \\ &\leq \int_1^{\infty}\left[\sum_{l=0}^{\infty}(2l+1)\left(\sum_{i=1}^{4}||V_{pq}f_{li}(t)||\right)^2\right]^{\frac{1}{2}} dt \leq \int_1^{\infty}\sum_{i=1}^{4}\left[\sum_{l=0}^{\infty}(2l+1)||V_{pq}f_{li}(t)||^2\right]^{\frac{1}{2}} dt \\ &= \text{const.}\int_1^{\infty}\sum_{i=1}^{4}[I_i(t)]^{\frac{1}{2}}\frac{1}{t}dt < \infty, \end{aligned} \right\} \tag{2.8.31}$$

and similarly for the interval $-\infty < t < -1$. This relation applies to all the two-body interactions V_{pq} contained in V_a. Hence, with eq. (2.8.17),

$$\sum_{l=0}^{\infty}(2l+1)\left[\int_{-\infty}^{\infty}||V_a e^{-iH_a t} f_{al}||dt\right]^2 < \infty. \tag{2.8.32}$$

There is a similar inequality for channel b. From this it follows with Schwarz's inequality that the right-hand side of eq. (2.8.11) is finite, as we wished to show. We can now use the relation (2.8.12), with the result that

$$\left.\begin{aligned}&\lim_{\varepsilon\to 0}\int \bar{g}(k_b)(V_b\varphi_b e^{i\boldsymbol{k}_b\cdot\boldsymbol{x}_b}, R(E+i\varepsilon)V_a\varphi_a e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a})\hat{f}(k_a)k_bk_a dE\\ &= 2\pi i\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)\lim_{X_b\to\infty}\int\bar{g}(k_b)[\delta_{ba}-S_{ba}(E,l)-\pi i(P_b(X_b)\chi_{bl},V_a\chi_{al})]\hat{f}(k_a)\sqrt{k_bk_a}dE,\end{aligned}\right\}\quad(2.8.33)$$

the series on the right being absolutely convergent.

2.8.3. The imaginary part of the scattering amplitude

If it is understood that φ_a and φ_b have been chosen real, there is a much more powerful result for the imaginary part of F_{ba}. Since the Hamiltonian commutes with the conjugation operator which transforms f into $\bar{f}$, it follows with eqs. (2.3.48) and (2.8.4) that

$$\left.\begin{aligned}\operatorname{Im}F_{ba}(E,\vartheta;\varepsilon) &= \lim_{X_b\to\infty}\operatorname{Im}F_{ba}(E,\vartheta;\varepsilon;X_b)\\ &= \frac{1}{2i}(V_b\varphi_b e^{i\boldsymbol{k}_b\cdot\boldsymbol{x}_b},[R(E+i\varepsilon)-R(E-i\varepsilon)]V_a\varphi_a e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a}).\end{aligned}\right\}\quad(2.8.34)$$

Also, by eq. (2.8.33),

$$\left.\begin{aligned}&\lim_{\varepsilon\to 0}\int\bar{g}(k_b)[\operatorname{Im}F_{ba}(E,\vartheta;\varepsilon)]\hat{f}(k_a)k_bk_a dE\\ &= \lim_{\varepsilon\to 0}2\pi\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)\int\bar{g}(k_b)[\delta_{ba}-\operatorname{Re}\tilde{S}_{ba}(E,l;\varepsilon;X_b)]\hat{f}(k_a)\sqrt{k_bk_a}\,dE\\ &= 2\pi\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)\int\bar{g}(k_b)[\delta_{ba}-\operatorname{Re}S_{ba}(E,l)]\hat{f}(k_a)\sqrt{k_bk_a}\,dE.\end{aligned}\right\}\quad(2.8.35)$$

Let us now first concentrate on the case $a = b$, and let us choose $\hat{f}$ and $\hat{g}$ positive. In each term of the series in the third member of eq. (2.8.35) the integrand is then positive, owing to eq. (2.7.63). The series converges absolutely, by our previous analysis. From this it follows that the series

$$2\pi\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)\bar{g}(k_a)[1-\operatorname{Re}S_{aa}(E,l)]\hat{f}(k_a)k_a \quad (2.8.36)$$

converges for almost every E, its sum being an integrable function (Burkill (22) section 3.10). Denoting the sum in question by

$$\bar{g}(k_a)[\operatorname{Im}F_{aa}(E,\vartheta)]\hat{f}(k_a)k_a^2 \quad (2.8.37)$$

we have

$$\lim_{\varepsilon\to 0}\int\bar{g}(k_a)[\operatorname{Im}F_{aa}(E,\vartheta;\varepsilon)]\hat{f}(k_a)k_a^2 dE = \int\bar{g}(k_a)[\operatorname{Im}F_{aa}(E,\vartheta)]\hat{f}(k_a)k_a^2 dE. \quad (2.8.38)$$

Now let $\hat{f}$ and $\hat{g}$ take the value 1 in a certain interval I, and let us consider the second member of eq. (2.8.35). This involves $R(E+i\varepsilon)-R(E-i\varepsilon)$, by analogy with eq. (2.8.34). If h is any function in $\mathfrak{L}^2$, we have

$$(h,[R(E+i\varepsilon)-R(E-i\varepsilon)]h) = 2i\varepsilon||R(E+i\varepsilon)h||^2. \tag{2.8.39}$$

As a result

$$1-\mathrm{Re}\tilde{S}_{aa}(E,l;\varepsilon;X_a) \geq 0. \tag{2.8.40}$$

Hence, $\hat{f}$ and $\hat{g}$ being positive,

$$\int_I [1-\mathrm{Re}\tilde{S}_{aa}(E,l;\varepsilon;X_a)]k_a dE \leq \int \bar{\hat{g}}(k_a)[1-\mathrm{Re}\tilde{S}_{aa}(E,l;\varepsilon;X_a)]\hat{f}(k_a)k_a dE. \tag{2.8.41}$$

Since we know from the previous section that the series in eq. (2.8.35) converge uniformly with respect to ε, it now follows that

$$\left.\begin{aligned}&\lim_{\varepsilon\to 0}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)\int_I[1-\mathrm{Re}\tilde{S}_{aa}(E,l;\varepsilon;X_a)]k_a dE\\ &=\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)\lim_{\varepsilon\to 0}\int_I[1-\mathrm{Re}\tilde{S}_{aa}(E,l;\varepsilon;X_a)]k_a dE,\end{aligned}\right\} \tag{2.8.42}$$

either series in eq. (2.8.42) converging absolutely, uniformly with respect to ε. On the right the limit with respect to ε can be performed with eq. (2.7.60). The argument which led to eq. (2.8.38) then yields

$$\lim_{\varepsilon\to 0}\int_I \mathrm{Im}F_{aa}(E,\vartheta;\varepsilon)k_a^2 dE = \int_I \mathrm{Im}\,F_{aa}(E,\vartheta)k_a^2 dE. \tag{2.8.43}$$

If E_1 and E_2 are any two points in the interval I, and the integral

$$\int_{E_1}^{E_2}\mathrm{Im}\,F_{aa}(E,\vartheta;\varepsilon)k_a^2 dE \tag{2.8.44}$$

is considered as a function of E_2, this function is of bounded variation, uniformly with respect to ε. If this result is combined with eq. (2.8.43), it follows from the Helly-Bray theorem on limits of Stieltjes integrals (Widder (30) ch. I, theorem 16.4) that eq. (2.8.38) holds true in all cases in which $\bar{\hat{g}}(k_a)\hat{f}(k_a)$ vanishes outside a bounded interval and is continuous except for a finite number of jumps.

To extend the foregoing to the case $a \neq b$, we observe that

$$|\delta_{ba}-\mathrm{Re}\tilde{S}_{ba}(E,l;\varepsilon;X_b)|^2 \leq [1-\mathrm{Re}\tilde{S}_{bb}(E,l;\varepsilon;X_b)][1-\mathrm{Re}\tilde{S}_{aa}(E,l;\varepsilon;X_b)], \tag{2.8.45}$$

owing to eq. (2.8.39). With the help of this inequality it is readily shown that there exists a function $\mathrm{Im}\,F_{ba}(E,\vartheta)$ which for almost every E satisfies

$$\mathrm{Im}\, F_{ba}(E,\vartheta)k_b k_a = 2\pi \sum_{l=0}^{\infty} (2l+1)P_l(\cos\vartheta)[\delta_{ba} - \mathrm{Re}S_{ba}(E,l)]\sqrt{k_b k_a}. \quad (2.8.46)$$

If $\bar{\hat{g}}(k_b)\hat{f}(k_a)$ vanishes outside a bounded region and is continuous except for a finite number of jumps, we have

$$\lim_{\varepsilon\to 0}\int \bar{\hat{g}}(k_b)[\mathrm{Im}F_{ba}(E,\vartheta;\varepsilon)]\hat{f}(k_a)k_b k_a dE = \int \bar{\hat{g}}(k_b)[\mathrm{Im}\, F_{ba}(E,\vartheta)]\hat{f}(k_a)k_b k_a dE. \quad (2.8.47)$$

The function $\mathrm{Im}F_{ba}(E,\vartheta)/4\pi$ is called the imaginary part of the scattering amplitude.

It is sometimes convenient to consider $\cos\vartheta$ as a function of E and

$$\Delta = |\boldsymbol{k}_a - \boldsymbol{k}_b|, \quad (2.8.48)$$

according to

$$\cos\vartheta = (k_a^2 + k_b^2 - \Delta^2)/2k_a k_b. \quad (2.8.49)$$

In order that $|\cos\vartheta| \leq 1$, it is necessary that

$$|k_a - k_b| \leq \Delta \leq k_a + k_b. \quad (2.8.50)$$

If this condition is fulfilled throughout the interval I, we have

$$\left.\begin{aligned} &\lim_{\varepsilon\to 0}\int_I P_l((k_a^2 + k_b^2 - \Delta^2)/2k_a k_b)[\delta_{ba} - \mathrm{Re}\tilde{S}_{ba}(E,l;\varepsilon;X_b)]\sqrt{k_b k_a}\, dE \\ &\quad = \int_I P_l((k_a^2 + k_b^2 - \Delta^2)/2k_a k_b)[\delta_{ba} - \mathrm{Re}S_{ba}(E,l)]\sqrt{k_b k_a}\, dE, \end{aligned}\right\} \quad (2.8.51)$$

owing to eq. (2.7.60). The convergence properties of sums of integrals of this form can be discussed with the methods developed above. It can also be shown that both eq. (2.8.46) and eq. (2.8.47) remain valid when $\cos\vartheta$ is considered to be function of E and Δ, provided Δ is such that $|\cos\vartheta| \leq 1$ throughout the energy region considered.

2.8.4. The scattering amplitude

To get some insight into the real part of F_{ba}, a more elaborate analysis is required. For this we assume as before that V_a satisfies the restriction imposed by eq. (2.8.18). It is sufficient if V_b is square-integrable. It is assumed that $\hat{f}(k_a)$ belongs to $\hat{\mathfrak{H}}_{lm}$. The function $\hat{g}(k_b)$ may be any function satisfying

$$\int |\hat{g}(k_b)|^2 k_b^2 dk_b < \infty. \quad (2.8.52)$$

We define f_{al} and g_{bl} as in eq. (2.8.8).

It follows from the method by which an expression for S_{ba} was obtained from eq. (2.6.1) that

$$\left.\begin{aligned} &|\tfrac{1}{2}\int \bar{g}(k_b)[\delta_{ba}-\tilde{S}_{ba}(E,l;\varepsilon;X_b)]\hat{f}(k_a)\sqrt{k_b k_a}\,dE| \\ &= |\int_{-\infty}^{\infty}(g_{bl},e^{iH_bt}[P_b(X_b)-1+\Omega^{*}_{b-,\,\varepsilon}]V_a e^{-iH_at}f_{al})dt| \leq 2||g_{bl}||\int_{-\infty}^{\infty}||V_a e^{-iH_at}f_{al}||dt \end{aligned}\right\} \quad (2.8.53)$$

(cf. eqs. (2.7.48) and (2.7.50)). Taking in particular

$$\hat{g}(k_b) = [\delta_{ba}-\tilde{S}_{ba}(E,l;\varepsilon;X_b)]\hat{f}(k_a)\sqrt{k_a/k_b}, \quad (2.8.54)$$

we obtain

$$\tfrac{1}{2}\int|[\delta_{ba}-\tilde{S}_{ba}(E,l;\varepsilon;X_b)]\hat{f}(k_a)|^2 k_a dE \leq 4\Big[\int_{-\infty}^{\infty}||V_a e^{-iH_at}f_{al}||dt\Big]^2. \quad (2.8.55)$$

Now, in virtue of eq. (2.7.60),

$$\left.\begin{aligned} &\{\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\}\{\lim_{\varepsilon'\to 0}\lim_{X_b'\to\infty}\}\int\bar{\hat{f}}(k_a)[\delta_{ba}-\bar{\tilde{S}}_{ba}(E,l;\varepsilon';X_b')][\delta_{ba}-\tilde{S}_{ba}(E,l;\varepsilon;X_b)]\hat{f}(k_a)k_a dE \\ &= \int|[\delta_{ba}-S_{ba}(E,l)]\hat{f}(k_a)|^2 k_a dE. \end{aligned}\right\} \quad (2.8.56)$$

Hence, with eq. (2.8.55),

$$\tfrac{1}{2}\int|[\delta_{ba}-S_{ba}(E,l)]\hat{f}(k_a)|^2 k_a dE \leq 4\Big[\int_{-\infty}^{\infty}||V_a e^{-iH_at}f_{al}||dt\Big]^2. \quad (2.8.57)$$

Owing to eq. (2.8.32) it now follows that

$$\int_{-1}^{1}d\cos\vartheta\int|\sum_{l=N}^{\infty}(2l+1)P_l(\cos\vartheta)[\delta_{ba}-\tilde{S}_{ba}(E,l;\varepsilon;X_b)]\hat{f}(k_a)|^2 k_a dE \quad (2.8.58)$$

tends to 0 as N tends to ∞, uniformly in ε, X_b. In view of eq. (2.8.7) this means that, given a positive ξ, we can determine N in such a way that

$$\int_{-1}^{1}d\cos\vartheta\int\left|\left\{\frac{\sqrt{k_b k_a}}{2\pi i}F_{ba}(E,\vartheta;\varepsilon;X_b)-\sum_{l=0}^{N}(2l+1)P_l(\cos\vartheta)[\delta_{ba}-\tilde{S}_{ba}(E,l;\varepsilon;X_b)]\right\}\hat{f}(k_a)\right|^2 k_a dE < \xi \quad (2.8.59)$$

for every ε, X_b. There also exists a function $F_{ba}(E,\vartheta)$ such that

$$\int_{-1}^{1}d\cos\vartheta\int\left|\left\{\frac{\sqrt{k_b k_a}}{2\pi i}F_{ba}(E,\vartheta)-\sum_{l=0}^{N}(2l+1)P_l(\cos\vartheta)[\delta_{ba}-S_{ba}(E,l)]\right\}\hat{f}(k_a)\right|^2 k_a dE < \xi, \quad (2.8.60)$$

by eqs. (2.8.32), (2.8.57) and the Riesz-Fischer theorem (Riesz and Sz.-Nagy (11) section 28). This function is square-integrable in the sense that

$$\int_{-1}^{1}d\cos\vartheta\int_{I}|F_{ba}(E,\vartheta)|^2 k_b k_a^2 dE < \infty, \quad (2.8.61)$$

where I may be any bounded interval contained in the interval (2.7.45). In the space of all functions satisfying eq. (2.8.61) we can say that

$$F_{ba}(E,\vartheta) = \underset{N\to\infty}{\text{l.i.m.}} \frac{2\pi i}{\sqrt{k_b k_a}} \sum_{l=0}^{N} (2l+1)P_l(\cos\vartheta)[\delta_{ba} - S_{ba}(E,l)]. \qquad (2.8.62)$$

In the following $F_{ba}(E,\vartheta)/4\pi$ is called the scattering amplitude. If a function $\operatorname{Im} F_{ba}(E,\vartheta)$ can be defined according to eq. (2.8.46), it is clear that this is the imaginary part of the present function $F_{ba}(E,\vartheta)$. We recall in this connection that in the previous section we obtained the result that the series in eq. (2.8.46) converges for almost every E. For this we had to assume that V_b satisfies eq. (2.8.18). In the present section we merely consider convergence in mean, which is established under the much weaker condition that V_b be square-integrable. In either section V_a is assumed to satisfy eq. (2.8.18).

In an obvious notation we can write eqs. (2.8.59) and (2.8.60) in the form

$$\left.\begin{aligned}\int_{-1}^{1} d\cos\vartheta \int |[F(E,\vartheta;\varepsilon;X_b) - F_N(E,\vartheta;\varepsilon;X_b)]\hat{f}(k_a)|^2 k_b k_a^2 dE &< 2\pi\xi,\\ \int_{-1}^{1} d\cos\vartheta \int |[F(E,\vartheta) - F_N(E,\vartheta)]\hat{f}(k_a)|^2 k_b k_a^2 dE &< 2\pi\xi.\end{aligned}\right\} \qquad (2.8.63)$$

Let us now consider the integral

$$\int_{-1}^{1} d\cos\vartheta \int \bar{B}(E,\vartheta) F_{ba}(E,\vartheta;\varepsilon;X_b)\hat{f}(k_a)\sqrt{k_b k_a}\, k_a dE, \qquad (2.8.64)$$

where B satisfies

$$\int_{-1}^{1} d\cos\vartheta \int |B(E,\vartheta)|^2 k_a dE < \infty. \qquad (2.8.65)$$

According to eq. (2.8.63) and Schwarz's inequality, there exists an integer N such that, given ξ,

$$\left.\begin{aligned}\left|\int_{-1}^{1} d\cos\vartheta \int \bar{B}(E,\vartheta)[F(E,\vartheta;\varepsilon;X_b) - F_N(E,\vartheta;\varepsilon;X_b)]\hat{f}(k_a)\sqrt{k_b k_a}\, k_a dE\right| &< \xi,\\ \left|\int_{-1}^{1} d\cos\vartheta \int \bar{B}(E,\vartheta)[F(E,\vartheta) - F_N(E,\vartheta)]\hat{f}(k_a)\sqrt{k_b k_a}\, k_a dE\right| &< \xi,\end{aligned}\right\} \qquad (2.8.66)$$

for every ε, X_b. Now it is not difficult to see that

$$\int_{-1}^{1} P_l(\cos\vartheta) B(E,\vartheta)\sqrt{k_a/k_b}\, d\cos\vartheta \qquad (2.8.67)$$

can be considered as a function $\hat{g}(k_b)$ satisfying eq. (2.8.52). Hence, given N, we can choose ε, X_b in such a way that

$$\left| \int_{-1}^{1} d\cos\vartheta \int \bar{B}(E,\vartheta)[F_N(E,\vartheta) - F_N(E,\vartheta;\varepsilon;X_b)]\hat{f}(k_a)\sqrt{k_b k_a}\, k_a dE \right| < \xi, \qquad (2.8.68)$$

owing to eq. (2.7.60). If this result is combined with eq. (2.8.66), it follows that

$$\left.\begin{aligned} &\{\lim_{\varepsilon\to 0} \lim_{X_b\to\infty}\} \int_{-1}^{1} d\cos\vartheta \int \bar{B}(E,\vartheta)F_{ba}(E,\vartheta;\varepsilon;X_b)\hat{f}(k_a)\sqrt{k_b k_a}\, k_a dE \\ &\quad = \int_{-1}^{1} d\cos\vartheta \int \bar{B}(E,\vartheta)F_{ba}(E,\vartheta)\hat{f}(k_a)\sqrt{k_b k_a}\, k_a dE. \end{aligned}\right\} \qquad (2.8.69)$$

In particular, since F satisfies eq. (2.8.61),

$$\left.\begin{aligned} &\{\lim_{\varepsilon\to 0} \lim_{X_b\to\infty}\}\{\lim_{\varepsilon'\to 0} \lim_{X'_b\to\infty}\} \int_{-1}^{1} d\cos\vartheta \int \bar{\hat{f}}(k_a)\bar{F}_{ba}(E,\vartheta;\varepsilon';X'_b)F_{ba}(E,\vartheta;\varepsilon;X_b)\hat{f}(k_a)k_b k_a^2 dE \\ &\quad = \int_{-1}^{1} d\cos\vartheta \int |F_{ba}(E,\vartheta)\hat{f}(k_a)|^2 k_b k_a^2 dE. \end{aligned}\right\} \qquad (2.8.70)$$

In the present section it has been assumed thus far that $\hat{f}$ belongs to $\hat{\mathfrak{G}}_{lm}$. However, given the fact that eqs. (2.8.63) and (2.8.66) hold true for functions $\hat{f}$ in $\hat{\mathfrak{G}}_{lm}$, it is clear that these equations are satisfied for every $\hat{f}$ which is bounded and vanishes outside a bounded region, it being understood that the integer N depends on the particular $\hat{f}$ considered. Also, since for eq. (2.7.60) to hold true it is sufficient if $\hat{f}$ belongs to $\mathfrak{S}\hat{\mathfrak{G}}_{lm}$, eq. (2.8.68) is valid for every $\hat{f}$ in $\mathfrak{S}\hat{\mathfrak{G}}_{lm}$. Hence so are eqs. (2.8.69) and (2.8.70).

Now let E_1 and E_2 be two points in some bounded interval I contained in the interval (2.7.45). If E_1 is held fixed and the integral

$$\int_{-1}^{1} d\cos\vartheta \int_{E_1}^{E_2} \bar{B}(E,\vartheta)F_{ba}(E,\vartheta;\varepsilon;X_b)\sqrt{k_b k_a}\, k_a dE \qquad (2.8.71)$$

is considered as a function of E_2, this function is of bounded variation, uniformly with respect to ε, X_b. This can be shown with Schwarz's inequality and the methods used in connection with eqs. (2.8.41) to (2.8.44). It follows with the Helly–Bray theorem that eqs. (2.8.69) and (2.8.70) hold true for every $\hat{f}$ which vanishes outside a bounded interval and is continuous except for a finite number of jumps.

It is easily checked that throughout this section $\cos\vartheta$ can be considered as a function of E and Δ, according to eq. (2.8.49).

2.8.5. Beams of projectiles

Let us consider a system which in the distant past was in channel a and behaved according to some wave-function $\exp(-iH_a t)\varphi_a(\boldsymbol{x}_a')f(\boldsymbol{x}_a)$. If the wave-function is decomposed into an incident wave plus a scattered wave, and it is assumed that V_a and V_b are square-integrable, the probability that in the remote future the scattered wave will be in channel b is given by

$$\left.\begin{aligned} ||(S_{ba}-\delta_{ba})f_a||^2 &= \tfrac{1}{4}(2\pi)^{-4}\{\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\}\{\lim_{\varepsilon'\to 0}\lim_{X_b'\to\infty}\}\int \bar{A}_{ba}(\boldsymbol{k}_b;\varepsilon';X_b')A_{ba}(\boldsymbol{k}_b;\varepsilon;X_b)d\boldsymbol{k}_b,\\ A_{ba}(\boldsymbol{k}_b;\varepsilon;X_b) &= \int k_a F_{ba}(\boldsymbol{k}_b,\boldsymbol{k}_a;\varepsilon;X_b)\hat{f}(\boldsymbol{k}_a)d\omega_{k_a}. \end{aligned}\right\} \quad (2.8.72)$$

This is a simplified form to which eq. (2.6.71) can be reduced if $m_a = m_b = 2$ and V_a and V_b are square-integrable.

In the initial wave-function as well as in the function F_{ba}, there occurs a vector $\boldsymbol{x}_a$. This denotes the distance between the two fragments which are scattered at each other. For the following it is convenient to consider the scattering in a coordinate frame in which one of the fragments, the target, is fixed at the origin. Obviously $\boldsymbol{x}_a$ then stands for the distance between the origin and the projectile. The motion of the projectile is determined by $f(\boldsymbol{x}_a)$.

We now compare the event considered in eq. (2.8.72) with the scattering from an initial state

$$e^{-iH_a t}f_a(\boldsymbol{r}) \equiv e^{-iH_a t}\varphi_a(\boldsymbol{x}_a')f(\boldsymbol{x}_a+\boldsymbol{r}). \quad (2.8.73)$$

In this state the motion of the projectile is determined by $f(\boldsymbol{x}_a+\boldsymbol{r})$, hence in the distant past the projectile behaved like the original one, except for a translation over $\boldsymbol{r}$. By analogy with eq. (2.8.72), we get

$$\left.\begin{aligned} ||(S_{ba}-\delta_{ba})f_a(\boldsymbol{r})||^2 &= \tfrac{1}{4}(2\pi)^{-4}\{\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\}\{\lim_{\varepsilon'\to 0}\lim_{X_b'\to\infty}\}\int \bar{A}_{ba}(\boldsymbol{k}_b;\boldsymbol{r};\varepsilon';X_b')A_{ba}(\boldsymbol{k}_b;\boldsymbol{r};\varepsilon;X_b)d\boldsymbol{k}_b,\\ A_{ba}(\boldsymbol{k}_b;\boldsymbol{r};\varepsilon;X_b) &= \int k_a F_{ba}(\boldsymbol{k}_b,\boldsymbol{k}_a;\varepsilon;X_b)e^{i\boldsymbol{k}_a\cdot\boldsymbol{r}}\hat{f}(\boldsymbol{k}_a)d\omega_{k_a}. \end{aligned}\right\} \quad (2.8.74)$$

Now let $\boldsymbol{r}$ be a two-component vector which varies over some plane ϱ. Let us consider a statistical mixture of projectiles $f(\boldsymbol{x}_a+\boldsymbol{r})$ in which the number of projectiles with $\boldsymbol{r}$-vector in $d\boldsymbol{r}$ is equal to $d\boldsymbol{r}$. In the following such a mixture is called a beam. We shall see below that within the framework of Hilbert space it provides a good description of what one usually tries to discuss in terms of plane waves. It will be understood that, if a beam is scattered, all the projectiles are scattered independently, i. e. the total scattering intensity is the integral over $\boldsymbol{r}$ of the intensities due to the separate projectiles. In channel b the beam $f(\boldsymbol{x}_a+\boldsymbol{r})$ thus yields a scattering intensity

$$I_{ba} = \int_{\varrho} ||(S_{ba}-\delta_{ba})f_a(\boldsymbol{r})||^2 d\boldsymbol{r}. \quad (2.8.75)$$

This quantity is now discussed under the assumption that V_b is square-integrable and that V_a satisfies the restriction imposed by eq. (2.8.18).

2.8.6. The scattering intensity

In virtue of eqs. (2.8.5) and (2.8.7)

$$\left.\begin{aligned} &\tfrac{1}{4}(2\pi)^{-4}\int \bar{A}_{ba}(\boldsymbol{k}_b;\boldsymbol{r};\varepsilon';X_b')A_{ba}(\boldsymbol{k}_b;\boldsymbol{r};\varepsilon;X_b)d\boldsymbol{k}_b \\ &= \tfrac{1}{2}\sum_{l'=0}^{\infty}\sum_{m'}\sum_{l=0}^{\infty}\sum_{m}\int \bar{Y}_{l'm'}(\omega_{k_b})Y_{lm}(\omega_{k_b})d\omega_{k_b}\int[\delta_{ba}-\bar{\hat{S}}_{ba}(E,l';\varepsilon';X_b')][\delta_{ba}-\hat{S}_{ba}(E,l;\varepsilon;X_b)]k_a dE \\ &\quad \times \int Y_{l'm'}(\omega_{k_a'})e^{-i\boldsymbol{k}_a'\cdot\boldsymbol{r}}\hat{f}(\boldsymbol{k}_a')d\omega_{k_a'}\int \bar{Y}_{lm}(\omega_{k_a})e^{i\boldsymbol{k}_a\cdot\boldsymbol{r}}\hat{f}(\boldsymbol{k}_a)d\omega_{k_a}. \end{aligned}\right\}\quad (2.8.76)$$

Here ω_{k_a} and $\omega_{k_a'}$ stand for the polar angles of certain vectors $\boldsymbol{k}_a$ and $\boldsymbol{k}_a'$ which are both of length k_a. If

$$\int |\hat{f}(\boldsymbol{k}_a)|^2 d\omega_{k_a} < M < \infty \quad (2.8.77)$$

and $\hat{f}(\boldsymbol{k}_a)$ vanishes if E is outside some bounded interval I, then each term of the series in eq. (2.8.76) tends to a limit as $\varepsilon, X_b, \varepsilon', X_b'$ tend to $0, \infty, 0, \infty$, owing to eq. (2.7.60). Taking into account that

$$\int |\bar{Y}_{l'm'}(\omega_{k_b})Y_{lm}(\omega_{k_b})|d\omega_{k_b} \le 1, \quad (2.8.78)$$

$$\sum_{l=0}^{\infty}\sum_{m}\left|\int \bar{Y}_{lm}(\omega_{k_a})e^{i\boldsymbol{k}_a\cdot\boldsymbol{r}}\hat{f}(\boldsymbol{k}_a)d\omega_{k_a}\right|^2 = \int |\hat{f}(\boldsymbol{k}_a)|^2 d\omega_{k_a} < M, \quad (2.8.79)$$

it follows from Schwarz's inequality that the series in eq. (2.8.76) is dominated by

$$\left.\begin{aligned} &\tfrac{1}{2}M\left[\sum_{l'=0}^{\infty}(2l'+1)\int_I |\delta_{ba}-\hat{S}_{ba}(E',l';\varepsilon';X_b')|^2 k_a' dE'\right]^{\frac{1}{2}} \\ &\times\left[\sum_{l=0}^{\infty}(2l+1)\int_I |\delta_{ba}-\hat{S}_{ba}(E,l;\varepsilon;X_b)|^2 k_a dE\right]^{\frac{1}{2}}. \end{aligned}\right\}\quad (2.8.80)$$

In this expression either series converges uniformly with respect to $\varepsilon, X_b, \varepsilon', X_b'$, owing to eqs. (2.8.32) and (2.8.55). As a result the limit with respect to $\varepsilon, X_b, \varepsilon', X_b'$ of the series in eq. (2.8.76) is the sum of the limits of the separate terms. Also,

$$\left.\begin{aligned} &\tfrac{1}{4}(2\pi)^{-4}\{\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\}\{\lim_{\varepsilon'\to 0}\lim_{X_b'\to\infty}\}\int \bar{A}_{ba}(\boldsymbol{k}_b;\boldsymbol{r};\varepsilon';X_b')A_{ba}(\boldsymbol{k}_b;\boldsymbol{r};\varepsilon;X_b)d\boldsymbol{k}_b \\ &= \tfrac{1}{2}\int d\omega_{k_b}\int\Big|\sum_{l=0}^{\infty}\sum_{m}Y_{lm}(\omega_{k_b})[\delta_{ba}-S_{ba}(E,l)]\int \bar{Y}_{lm}(\omega_{k_a})e^{i\boldsymbol{k}_a\cdot\boldsymbol{r}}\hat{f}(\boldsymbol{k}_a)d\omega_{k_a}\Big|^2 k_a dE. \end{aligned}\right\}\quad (2.8.81)$$

Extending the integration with respect to ω_{k_b} over the full angle 4π, we obtain, with eq. (2.8.74),

$$||(S_{ba}-\delta_{ba})f_a(\boldsymbol{r})||^2 = \tfrac{1}{2}\sum_{l=0}^{\infty}\sum_m \int |[\delta_{ba}-S_{ba}(E,l)]\int \bar{Y}_{lm}(\omega_{k_a})e^{i\boldsymbol{k}_a\cdot\boldsymbol{r}}\hat{f}(\boldsymbol{k}_a)d\omega_{k_a}|^2 k_a dE. \quad (2.8.82)$$

At this point it is convenient to introduce a rectangular coordinate frame with axes 1 and 2 in the plane ϱ and axis 3 perpendicular to it. Writing

$$\left.\begin{aligned} k_{a1} &= k_a \sin\beta_a \cos\alpha_a, \\ k_{a2} &= k_a \sin\beta_a \sin\alpha_a, \\ k_{a3} &= k_a \cos\beta_a, \end{aligned}\right\} \quad (2.8.83)$$

we obviously have

$$\int d\omega_{k_a} = \int d\cos\beta_a d\alpha_a. \quad (2.8.84)$$

Now let our beam be directed in the sense that $\hat{f}(\boldsymbol{k}_a) = 0$ if $\cos\beta_a < 0$. In the region of integration there is then a one-to-one correspondence between k_a,β_a,α_a and k_a, k_{a1},k_{a2}. Also,

$$\int d\cos\beta_a d\alpha_a = \int k_a^{-1}(k_a^2-k_{a1}^2-k_{a2}^2)^{-\frac{1}{2}}dk_{a1}dk_{a2}. \quad (2.8.85)$$

Hence

$$g_{lm}(E,\boldsymbol{r}) \equiv \int \bar{Y}_{lm}(\omega_{k_a})e^{i\boldsymbol{k}_a\cdot\boldsymbol{r}}\hat{f}(\boldsymbol{k}_a)d\omega_{k_a} \quad (2.8.86)$$

is the Fourier transform of

$$\hat{g}_{lm}(E,k_{a1},k_{a2}) \equiv 2\pi\bar{Y}_{lm}(\omega_{k_a})\hat{f}(\boldsymbol{k}_a)k_a^{-1}(k_a^2-k_{a1}^2-k_{a2}^2)^{-\frac{1}{2}}. \quad (2.8.87)$$

As a result

$$\left.\begin{aligned} \int_\varrho |g_{lm}(E,\boldsymbol{r})|^2 d\boldsymbol{r} &= \int |\hat{g}_{lm}(E,k_{a1},k_{a2})|^2 dk_{a1}dk_{a2} \\ &= 4\pi^2 k_a^{-2}\int |Y_{lm}(\omega_{k_a})\hat{f}(\boldsymbol{k}_a)|^2(\cos\beta_a)^{-1}d\cos\beta_a d\alpha_a. \end{aligned}\right\} \quad (2.8.88)$$

We now assume that

$$k_a^{-2}\int |\hat{f}(\boldsymbol{k}_a)|^2(\cos\beta_a)^{-1}d\cos\beta_a d\alpha_a < \infty, \quad (2.8.89)$$

a condition which implies eq. (2.8.77). If it is satisfied, it follows with eqs. (2.8.5) and (2.8.75) that

$$I_{ba} = \tfrac{1}{2}\pi\sum_{l=0}^{\infty}(2l+1)\int |\delta_{ba}-S_{ba}(E,l)|^2 k_a^{-1}dE\int |\hat{f}(\boldsymbol{k}_a)|^2(\cos\beta_a)^{-1}d\cos\beta_a d\alpha_a. \quad (2.8.90)$$

According to eq. (2.8.62), we may also write

$$I_{ba} = \frac{1}{16\pi}\int_{-1}^{1} d\cos\vartheta\int |F_{ba}(E,\vartheta)|^2 k_b dE\int |\hat{f}(\boldsymbol{k}_a)|^2(\cos\beta_a)^{-1}d\cos\beta_a d\alpha_a. \quad (2.8.91)$$

Alternatively, defining

$$F_{ba}(\boldsymbol{k}_a, \boldsymbol{k}_a) = F_{ba}(E, \vartheta) \tag{2.8.92}$$

we obtain

$$I_{ba} = \int I_{ba}(\omega_{k_b}) d\omega_{k_b} = \frac{1}{16\pi^2} \int d\boldsymbol{k}_b \int |F_{ba}(\boldsymbol{k}_b, \boldsymbol{k}_a) \hat{f}(\boldsymbol{k}_a)|^2 (\cos\beta_a)^{-1} d\cos\beta_a d\alpha_a. \tag{2.8.93}$$

2.8.7. Scattering in a fixed direction

Equation (2.8.93) suggests that $\int_{\delta\omega} I_{ba}(\omega_{k_b}) d\omega_{k_b}$ is the intensity of the scattering into the angle $\delta\omega$. That this is correct can be seen as follows. If in eq. (2.8.81) the integration over ω_{k_b} is restricted to $\delta\omega$, we obtain the probability that the projectile $f(\boldsymbol{x}_a + \boldsymbol{r})$ yields a wave scattered into the angle $\delta\omega$. This is obvious from the proof of eq. (2.6.71). Let us now define

$$\left.\begin{aligned} F_N(\boldsymbol{k}_b, \boldsymbol{k}_a) &= F_N(E, \vartheta), \\ A_N(\boldsymbol{k}_b, \boldsymbol{r}) &= \int k_a F_N(\boldsymbol{k}_b, \boldsymbol{k}_a) e^{i\boldsymbol{k}_a \cdot \boldsymbol{r}} \hat{f}(\boldsymbol{k}_a) d\omega_{k_a}, \end{aligned}\right\} \tag{2.8.94}$$

$F_N(E,\vartheta)$ being the function considered in eq. (2.8.63). Then it follows from eq. (2.8.63) that there is a function $A(\boldsymbol{k}_b,\boldsymbol{r})$ such that

$$\lim_{N\to\infty} \int |A(\boldsymbol{k}_b, \boldsymbol{r}) - A_N(\boldsymbol{k}_b, \boldsymbol{r})|^2 d\boldsymbol{k}_b = 0. \tag{2.8.95}$$

Either side of eq. (2.8.81) is equal to

$$\tfrac{1}{4}(2\pi)^{-4} \int |A(\boldsymbol{k}_b, \boldsymbol{r})|^2 d\boldsymbol{k}_b = \tfrac{1}{4}(2\pi)^{-4} \lim_{N\to\infty} \int |A_N(\boldsymbol{k}_b, \boldsymbol{r})|^2 d\boldsymbol{k}_b. \tag{2.8.96}$$

The intensity scattered into channel b is obtained from this expression by integrating over $\boldsymbol{r}$. Now

$$\left.\begin{aligned} \tfrac{1}{4}(2\pi)^{-4} \int_{\delta\omega} d\omega_{k_b} \int |A_N(\boldsymbol{k}_b, \boldsymbol{r})|^2 k_b^2 dk_b &\le \tfrac{1}{4}(2\pi)^{-4} \int_{4\pi} d\omega_{k_b} \int |A_N(\boldsymbol{k}_b, \boldsymbol{r})|^2 k_b^2 dk_b \\ &\le ||(S_{ba} - \delta_{ba}) f_a(\boldsymbol{r})||^2, \end{aligned}\right\} \tag{2.8.97}$$

the second inequality following with eq. (2.8.82). Since the right-hand side of eq. (2.8.97) is an integrable function of $\boldsymbol{r}$, it follows that, when integrating the expression (2.8.96) over $\boldsymbol{r}$, we have

$$\int d\boldsymbol{r} \lim_{N\to\infty} = \lim_{N\to\infty} \int d\boldsymbol{r}. \tag{2.8.98}$$

By analogy with eq. (2.8.88)

$$\int_{\Omega} |A_N(\boldsymbol{k}_b,\boldsymbol{r})|^2 d\boldsymbol{r} = 4\pi^2 \int |F_N(\boldsymbol{k}_b,\boldsymbol{k}_a)\hat{f}(\boldsymbol{k}_a)|^2 (\cos\beta_a)^{-1} d\cos\beta_a d\alpha_a. \qquad (2.8.99)$$

Owing to eq. (2.8.98), the scattering intensity in the angle $\delta\omega$ thus takes the form

$$\frac{1}{16\pi^2} \lim_{N\to\infty} \int_{\delta\omega} d\omega_{k_b} \int k_b^2 dk_b \int |F_N(\boldsymbol{k}_b,\boldsymbol{k}_a)\hat{f}(\boldsymbol{k}_a)|^2 (\cos\beta_a)^{-1} d\cos\beta_a d\alpha_a. \qquad (2.8.100)$$

The limit can now be performed with eq. (2.8.63). We simply obtain

$$\lim_{N\to\infty} \int_{\delta\omega} d\omega_{k_b} \int k_b^2 dk_b \int |F_N \hat{f}|^2 (\cos\beta_a)^{-1} d\omega_{k_a} = \int_{\delta\omega} d\omega_{k_b} \int k_b^2 dk_b \int |F\hat{f}|^2 (\cos\beta_a)^{-1} d\omega_{k_a}. \qquad (2.8.101)$$

This shows that $\int_{\delta\omega} I_{ba}(\omega_{k_b}) d\omega_{k_b}$ is the intensity of the scattering into the angle $\delta\omega$, as we wished to prove.

2.8.8. The cross section

To illustrate the physics of eq. (2.8.93), we consider the particular case that $\hat{f}(\boldsymbol{k}_a)$ vanishes except in a small region $d\boldsymbol{k}_a$. Decomposing our wave-functions into incident and scattered waves, we evaluate the total number of projectiles incident upon a surface element δs of a plane perpendicular to $\boldsymbol{k}_a$. This quantity is denoted by $\delta s \int N(\boldsymbol{k}_a) d\boldsymbol{k}_a$.

The incident wave associated with the projectile $f(\boldsymbol{x}_a + \boldsymbol{r})$ contains as a factor the relative motion

$$f(\boldsymbol{x}_a + \boldsymbol{r}, t) = (2\pi)^{-\frac{3}{2}} \int \exp[i(-k_a^2 t + \boldsymbol{k}_a\cdot\boldsymbol{x}_a + \boldsymbol{k}_a\cdot\boldsymbol{r})]\hat{f}(\boldsymbol{k}_a) d\boldsymbol{k}_a. \qquad (2.8.102)$$

This satisfies the Schrödinger equation

$$\left[\Delta(\boldsymbol{x}_a) + i\frac{\partial}{\partial t}\right] f(\boldsymbol{x}_a + \boldsymbol{r}, t) = 0. \qquad (2.8.103)$$

Hence there is a continuity equation of the form

$$\frac{\partial}{\partial t}|f(\boldsymbol{x}_a + \boldsymbol{r}, t)|^2 + \operatorname{div}\{\mathrm{Re}[-2i\bar{f}(\boldsymbol{x}_a + \boldsymbol{r}, t)\operatorname{grad} f(\boldsymbol{x}_a + \boldsymbol{r}, t)]\} = 0, \qquad (2.8.104)$$

the expression in curly brackets being a flux vector. The unconventional factor 2 is due to our normalization of $\boldsymbol{x}_a$ (cf. eq. (2.1.2)). With this factor, the number of projectiles $f(\boldsymbol{x}_a + \boldsymbol{r})$ with $\boldsymbol{r}$-vector in $d\boldsymbol{r}$ that pass through δs in the time interval dt takes the form

$$\left.\begin{aligned} &2(2\pi)^{-3}dtd\boldsymbol{r}\mathrm{Re}\Big[\int_{\delta s} d\boldsymbol{x}_a \int \exp[-i(-k_a'^2 t + \boldsymbol{k}_a'\cdot\boldsymbol{x}_a + \boldsymbol{k}_a'\cdot\boldsymbol{r})]\bar{\hat{f}}(\boldsymbol{k}_a')d\boldsymbol{k}_a' \\ &\qquad \times \int k_a \exp[i(-k_a^2 t + \boldsymbol{k}_a\cdot\boldsymbol{x}_a + \boldsymbol{k}_a\cdot\boldsymbol{r})]\hat{f}(\boldsymbol{k}_a)d\boldsymbol{k}_a\Big]. \end{aligned}\right\} \quad (2.8.105)$$

The total number of projectiles passing through δs at some time t is obtained from this expression by integrating over t and $\boldsymbol{r}$. If we remember that

$$\int d\boldsymbol{k}_a = \tfrac{1}{2}\int (k_a^2 - k_{a1}^2 - k_{a2}^2)^{-\frac{1}{2}} dk_a^2 dk_{a1} dk_{a2}, \quad (2.8.106)$$

the integration can be performed explicitly with the standard theory of Fourier transforms. The final result is

$$N(\boldsymbol{k}_a) = |\hat{f}(\boldsymbol{k}_a)|^2(\cos\beta_a)^{-1}. \quad (2.8.107)$$

Hence, according to eq. (2.8.93),

$$I_{ba}(\omega_{k_b}) = (4\pi)^{-2}\int |F_{ba}(\boldsymbol{k}_b, \boldsymbol{k}_a)|^2 N(\boldsymbol{k}_a) k_b k_a^{-1} d\boldsymbol{k}_a. \quad (2.8.108)$$

The function $|F_{ba}/4\pi|^2$ thus transforms the number of projectiles incident per unit area into the number scattered into a unit angle. It is the cross section for scattering from channel a into channel b.

2.8.9. The optical theorem

According to eq. (2.3.65) the total intensity of the scattering from channel a is given by

$$I = 2\int_{\varrho} \mathrm{Re}(f_a(\boldsymbol{r}), [1 - S_{aa}]f_a(\boldsymbol{r}))d\boldsymbol{r}. \quad (2.8.109)$$

With eq. (2.6.67) this reduces to

$$\left.\begin{aligned} I = \frac{1}{8\pi^2}\int d\boldsymbol{r}\,\mathrm{Im}\Big[\lim_{\varepsilon\to 0}\lim_{X_a\to\infty}\int dE \iint d\omega_{k_a'} d\omega_{k_a} k_a^2 \bar{\hat{f}}(\boldsymbol{k}_a') e^{-i\boldsymbol{k}_a'\cdot\boldsymbol{r}} \\ \times F_{aa}(\boldsymbol{k}_a', \boldsymbol{k}_a; \varepsilon; X_a)\hat{f}(\boldsymbol{k}_a)e^{i\boldsymbol{k}_a\cdot\boldsymbol{r}}\Big], \end{aligned}\right\} \quad (2.8.110)$$

where F_{aa} is the function (2.8.1) and $\boldsymbol{k}_a', \boldsymbol{k}_a$ are two vectors of length k_a. From the form of F_{aa} it is easily seen that

$$\left.\begin{aligned} I = -\frac{i}{16\pi^2}\int_{\varrho} d\boldsymbol{r}\lim_{\varepsilon\to 0}\int dE \iint d\omega_{k_a'} d\omega_{k_a} k_a^2 \bar{\hat{f}}(\boldsymbol{k}_a') e^{-i\boldsymbol{k}_a'\cdot\boldsymbol{r}} \\ \times (V_a\varphi_a e^{i\boldsymbol{k}_a'\cdot\boldsymbol{x}_a}, [R(E + i\varepsilon) - R(E - i\varepsilon)]V_a\varphi_a e^{i\boldsymbol{k}_a\cdot\boldsymbol{x}_a})\hat{f}(\boldsymbol{k}_a)e^{i\boldsymbol{k}_a\cdot\boldsymbol{r}}. \end{aligned}\right\} \quad (2.8.111)$$

With the help of eqs. (2.8.5) and (2.8.7), I can be developed according to

$$I = \int_{\varrho} d\boldsymbol{r} \lim_{\varepsilon\to 0} \sum_{l=0}^{\infty} \sum_{m} \int [1 - \mathrm{Re}\check{S}_{aa}(E,l;\varepsilon;X_a)] \left| \int \bar{Y}_{lm}(\omega_{k_a}) e^{i\boldsymbol{k}_a\cdot\boldsymbol{r}} \hat{f}(\boldsymbol{k}_a) d\omega_{k_a} \right|^2 k_a dE. \quad (2.8.112)$$

Now let $\hat{f}(\boldsymbol{k}_a)$ satisfy eq. (2.8.77) and let it vanish outside some bounded interval. Then each term of the series in eq. (2.8.112) tends to a limit as ε tends to 0, by eq. (2.7.60). Also, the function $g_{lm}(E,\boldsymbol{r})$ defined in eq. (2.8.86) is bounded uniformly with respect to l,m. The terms of the series in eq. (2.8.112) are all non-negative, by eq. (2.8.40). Therefore, since in eq. (2.8.42) the series on the left converges uniformly with respect to ε, so does the series in eq. (2.8.112). From this it follows that the limit of the sum is the sum of the limits of the separate terms. In other words,

$$I = \int_{\varrho} d\boldsymbol{r} \sum_{l=0}^{\infty} \sum_{m} \int [1 - \mathrm{Re}S_{aa}(E,l)] |g_{lm}(E,\boldsymbol{r})|^2 k_a dE. \quad (2.8.113)$$

If $\hat{f}(\boldsymbol{k}_a)$ satisfies the further restriction (2.8.89), the integration over $\boldsymbol{r}$ can be performed with the help of eq. (2.8.88). With eqs. (2.8.36) and (2.8.37), the result is

$$\left.\begin{aligned} I &= \pi \sum_{l=0}^{\infty} (2l+1) \int [1 - \mathrm{Re}S_{aa}(E,l)] k_a^{-1} dE \int |\hat{f}(\boldsymbol{k}_a)|^2 (\cos\beta_a)^{-1} d\cos\beta_a\, d\alpha_a \\ &= \int \mathrm{Im}\, F_{aa}(E,0) |\hat{f}(\boldsymbol{k}_a)|^2 (k_a \cos\beta_a)^{-1} d\boldsymbol{k}_a. \end{aligned}\right\} \quad (2.8.114)$$

By analogy with eq. (2.8.108) we may write

$$I = \int \mathrm{Im}\, F_{aa}(E,0) k_a^{-1} N(\boldsymbol{k}_a) d\boldsymbol{k}_a. \quad (2.8.115)$$

This shows that $\mathrm{Im}F_{aa}(E,0)k_a^{-1}$ is the total cross section, which is thus equal to $4\pi k_a^{-1}$ times the imaginary part of the forward elastic scattering amplitude. This is a special form of the optical theorem discussed in section 2.3.7.

2.8.10. Discussion

In the standard treatment of one-channel potential scattering, it is shown that an incoming plane wave $\exp(i\boldsymbol{k}_a\cdot\boldsymbol{x}_a)$ yields a radially outgoing wave which for large x_a behaves asymptotically as

$$\frac{1}{x_a} X_{aa}\left(\frac{k_a}{x_a}\boldsymbol{x}_a, \boldsymbol{k}_a\right) \exp(ik_a x_a), \quad (2.8.116)$$

with some function X_{aa} depending only on k_a and on the angle between $\boldsymbol{x}_a$ and $\boldsymbol{k}_a$. From the asymptotic behaviour it follows that $|X_{aa}|^2$ is the cross section for elastic scattering. If we compare the standard theory with the present formalism, we see that X_{aa} is nothing but $F_{aa}/4\pi$. Hence the name scattering amplitude used in the foregoing.

Since a plane wave is not of finite norm, it cannot easily be incorporated in our Hilbert-space formalism. It is, of course, possible to consider superpositions of plane waves, i. e. wave-packets. However, the expression we found for the scattering intensity due to a single wave-packet is more complicated than a mere integral of the form

$$\int d\boldsymbol{k}_b \int |F_{ba}(\boldsymbol{k}_b, \boldsymbol{k}_a)\hat{h}(\boldsymbol{k}_a)|^2 d\omega_{k_a}. \tag{2.8.117}$$

Indeed, in eqs. (2.6.71) and (2.8.74) we have multiple integrals

$$\int d\boldsymbol{k}_b \int \bar{F}_{ba}(\boldsymbol{k}_b, \boldsymbol{k}_a')\bar{\hat{h}}(\boldsymbol{k}_a')d\omega_{k_a'} \int F_{ba}(\boldsymbol{k}_b, \boldsymbol{k}_a)\hat{h}(\boldsymbol{k}_a)d\omega_{k_a}. \tag{2.8.118}$$

These apply to a single scattering event. What one observes experimentally is the cross section. This is the number of fragments emerging per unit angle per unit time, divided by the number incident per unit area per unit time. It thus refers to a stream of projectiles. One usually describes such a stream with the help of a plane wave, but in the present paper we use a statistical mixture of wave-packets called a beam. This procedure leads to the conditions (2.8.89) and (2.8.18) on the wave-function $\hat{f}(\boldsymbol{k}_a)$ and on the interaction, respectively. If these are fulfilled, the intensity scattered from the beam can be evaluated in a completely straightforward manner. It yields an expression of the form (2.8.117). From this it then follows that $|F_{ba}/4\pi|^2$ is the cross section. If the theory is set up in this way, the difficulties of the standard plane-wave theory are avoided completely. In particular, there are no normalization problems, nor is there any ambiguity as regards the channel concept. Also, if $\hat{f}(\boldsymbol{k}_a)$ vanishes except in a small region $d\boldsymbol{k}_a$, the beam $\hat{f}(\boldsymbol{k}_a)$ more closely resembles a collimated stream of projectiles than does the plane wave $\exp(i\boldsymbol{k}_a \cdot \boldsymbol{x}_a)$. To describe a scattering experiment, a beam as defined here is therefore an improvement on a plane wave.

The condition (2.8.89) implies that $\hat{f}(\boldsymbol{k}_a)$ must vanish as $\cos\beta_a = 0$. It thus guarantees that the beam properly passes through the plane ϱ. If $\hat{f}(\boldsymbol{k}_a)$ were different from 0 only in the neighbourhood of $\cos\beta_a = 0$, the beam would propagate almost parallel to the plane ϱ. Since all projectiles would then have almost the same interaction with the target, the total scattering intensity would not remain finite. Similarly, the scattering intensities due to the separate projectiles would add up to infinity if at large distances the interaction did not fall off sufficiently rapidly. It is only if the interaction tends to 0 reasonably fast that projectiles with large $\boldsymbol{r}$-values are not disturbed appreciably. Hence it is only under some suitable condition on V_{pq} that we may expect the integral over $\boldsymbol{r}$ of the separate scattering intensities to be convergent. A sufficient condition is formulated explicitly in eq. (2.8.18).

Throughout the present paper we have tried to keep the formalism mathematically rigorous. It is unfortunate that this has led to tedious considerations, particularly as regards limits with respect to ε. It must be emphasized, however, that every time we prove that there is a limit, the argument can be traced back to the time development in a scattering system, and to the properties of the interaction between scattered fragments. We thus feel that the limiting behaviour as ε tends to 0 reflects in a mathematical form what is the essence of a scattering event.

In eq. (2.7.44) the function $S_{ba}(E,l)$ is defined as the derivative of the limit of a sequence of integrals,

$$S_{ba}(E,l)k_a = \frac{d}{dE}\lim_{\varepsilon\to 0}\lim_{X_b\to\infty}\int^{E} S_{ba}(E',l;\varepsilon;X_b)k_a' dE'. \tag{2.8.119}$$

There is a similar relation for the imaginary part of the scattering amplitude, the definition of the real part being slightly more complicated. Now it is conceivable that in some, or perhaps even in many, cases we have

$$S_{ba}(E,l) = \lim_{\varepsilon\to 0}\lim_{X_b\to\infty} S_{ba}(E,l;\varepsilon;X_b). \tag{2.8.120}$$

However, thus far we have found no evidence to this effect. On the other hand, there is a much more useful limiting relation. In fact, it will be shown in a sequel to the present paper that for a large class of interactions there exists an analytic function $S_{ba}(E+i\varepsilon,l;X_b)$ such that

$$S_{ba}(E,l) = \lim_{\varepsilon\to 0}\lim_{X_b\to\infty} S_{ba}(E+i\varepsilon,l;X_b) \tag{2.8.121}$$

for almost every E. The scattering amplitude is likewise the boundary value of an analytic function. In the neighbourhood of the real axis this function is sufficiently smooth to satisfy a dispersion relation. Also, in virtue of eq. (2.8.121), there is a parameter expansion for the scattering matrix which brings out the existence of resonances against a smoothly varying background. The present results thus make a starting-point for further work. A sequel to this paper will show again that there is an intimate connection between the limiting behaviour of scattering functions and the qualitative features of scattering events.

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Note added in proof

After the present paper was completed we learned that FADDEEV (31, 32) has shown that in a three-particle system there is unitarity in the sense of eq. (2.3.35) under fairly mild conditions on the interaction. This result gives a partial answer to the problem discussed in sections 2.3.4 and 2.3.5.

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